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Controllability of the multi-agent system modeled by the threshold graph with one repeated degree

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ABSTRACT

The controllability issue of multi-agent systems modeled by a special class of graphs is studied. Suppose the modeling graph is simple, connected and driven by only one controller. The author shows that the system is controllable if the graph has exactly two vertices with the same degree and either one of these two vertices is selected to receive the control input. This result is extended to the case of the threshold graph that has only one repeated degree. Specifically, it is shown that if the threshold graph has only one repeated degree and its multiplicity is m, then $m-1$ is the minimum number of the controllers required to make the system controllable. In addition, the necessary and sufficient condition that characterizes the binary control vectors to ensure the controllability of the system is derived.

1. Introduction

Control of multi-agent systems can be seen as one of the most important research topics in the control area in the last decade. With the progresses of modern communication techniques and the availability of various cost-efficient hardware, constructing a multi-agent system becomes an important approach to solving many modern engineering problems. Examples include wireless sensor surveillance networks, power grid monitoring systems, and drone swarm systems. These systems feature the decentralization, modularization, scalability, and even mobility, which make them capable of undertaking a wide range of tasks. However, the functionality of a multi-agent system cannot be high if its control mechanisms on cooperation and information sharing and/or synchronization are far from being optimal. The importance of these issues triggers many research efforts in essential topics such as coordination control [1], formation control [2], distribution control [3,4] and so on [5,6]. In fact, the research on the control of multi-agent system also motivates the studies on the new approaches to checking some essential properties of the system. Determining the controllability, among others, is one of the topics that attract the most attentions. In modeling a multi-agent system, the conventional way is to use a graph with its vertex set and edge set characterizing how the agents interact with each other. This graph-theoretical model actually serves as one of the most important tools in analyzing networked systems. Though different networks have different properties, leading to various graph-theoretical parameters, we focus our studies on the controllability analysis under the mild conditions of linearity, time-invariance and agent consensus policy. In this framework the evolution of the system can be shown to follow the Laplacian dynamics. Its controllability can then be checked using the classical rank test by Kalman. However, in the case of complex systems that have many agents, the numerical precision issue renders the test unreliable and motivates the development of other approaches that do not need to check the rank. Current results in the literature include two categories. Those in the first one gives algebraic or algorithmic criteria as the sufficient conditions for the system uncontrollability [7–10]. Their approaches bottom on the fundamental reasoning that the structural symmetry, in certain sense, leads to the system uncontrollability. Graph-partitioning schemes are then designed to catch the specific symmetry property that makes the system uncontrollable. Nevertheless, if the conditions are not satisfied, the system controllability becomes inconclusive. The results in the second category are the controllability conclusions on the systems whose modeling graphs possess very special structures such as the path, grid, cycle, and the circulant network [11–13]. Since many engineering systems can actually be modeled by these graphs, the results in the second category are valuable in the design of controllable systems. However, the known controllable graph structures are very limited and thus it is very challenging to design a graph that is controllable and has some nice graph-theoretic properties simultaneously. For example, the algebraic connectivity, or the second smallest eigenvalue of the Laplacian matrix corresponding to the graph, is an important index of the robustness and synchronization of the graph [14,15]. It

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is also the dominating eigenvalue that determines the speed of zero input response of the graph. Among the known controllable graph structures mentioned above, their algebraic connectivity in general decreases as the number of vertices increases. Considering the risks of power and/or communication failures that lead to the shutdown of the controller, a graph structure whose algebraic connectivity is lower bounded by a positive constant is sometimes necessary.

In this paper we aim to expand the known controllable classes of the multi-agent systems and in particular to explore the modeling graphs that are controllable and have algebraic connectivity independent of the vertex number. We start from the single-controller case and consider the modeling graphs that have exactly two vertices with the same numbers of connection edges, or called with the same degree. Under the standard assumptions that the graphs are simple and connected, we show that if a k-vertex graph has exactly two vertices with the same degree, the repeated degree is the integer part of $k/2$. More importantly, we show that the system modeled by this particular graph is controllable if either one of the two vertices with the same degree is selected to receive the control signals. Our arguments are based on the fundamental matrix operations only and no advanced results from the graph theory are involved. It is well known that for a simple and connected graph, at least two edges have the same degree. If a simple and connected graph has exactly two vertices with the same degree, then it belongs to a particular class known as the threshold graphs. A very elegant result, due to Merris [16], concerning this class of graphs is that the algebraic connectivity is at least 1, independent of the vertex number. We thus extend the result, as our main contribution, from the single-controller case above to the multi-controller case where the system is modeled by a threshold graph with only one repeated degree and the multiplicity is m. The necessary and sufficient condition that governs the control vectors for the controllability of the system is derived. A straightforward implication by the condition is that $m-1$ is the minimum number of controllers required to make the system controllable. One of the simplest way to ensure the controllability is to add $m-1$ edges connecting the $m-1$ controllers to any $m-1$ of the m vertices with the same degree.

The rest of this paper is organized as follows. In Section 2 the framework of multi-agent systems is reviewed. The existing model and the controllability condition derived in the context of linear time-invariant systems evolving in the continuous-time domain are recapitulated. In Section 3 the main results on the controllability issues of the systems modeled with our special class of graphs are presented. The paper is concluded in Section 4, where interesting future topics are discussed as well.

2. Preliminaries

2.1. Framework

We first define several standard notations to be used later. For natural numbers k_1, k_2, we write $\mathbb{R}^{k_1\times k_2}$ as the $k_1 \times k_2$-dimensional real space. If k_1 or k_2 is 1, the simpler notation such as $\mathbb{R}^{k_2}$ or $\mathbb{R}^{k_1}$ is used. I_k and $O_k \in \mathbb{R}^{k\times k}$ represent the identity and zero matrices, respectively, of order k. $\mathbf{1}_k$ and $\mathbf{0}_k \in \mathbb{R}^k$ are the column vectors of 1's and 0's respectively. Occasionally we will drop the k for simplicity as the context is clear. A row-switching elementary matrix is formed by an identity matrix with its two rows switched. A permutation matrix is the product of row-switching elementary matrices. If J is a permutation matrix, then clearly J is an orthogonal matrix, namely, $JJ^T = I$. Suppose $P \in \mathbb{R}^{n\times n}$. $\lambda \in \mathbb{R}$ and $\upsilon \in \mathbb{R}^n$, we say $(\lambda, \boldsymbol{v})$ is an eigenpair of P if $P\boldsymbol{v} = \lambda\boldsymbol{v}$. For the matrix A, $A_{i\cdot}$ and $A_{\cdot j}$ are the ith row and jth column of A, respectively. We use $\mathrm{diag}(d_1, d_2, \ldots, d_k)$ to represent the diagonal matrix with diagonal entries $d_i, i \in \{1, 2, \ldots, k\}$. $R_{ij}^{(k)}$ is the elementary matrix such that $R_{i,j}^{(k)}A$ is to replace $A_{j\cdot}$ with $A_{j\cdot} + kA_{i\cdot}$. For $c \geq 0$, $\lfloor c\rfloor$ is the integer part of c and $\lceil c\rceil$ means the smallest integer greater than or equal to c. Also, $\hat{c}$ and $\check{c}$ are used to represent $\lceil \frac{c}{2}\rceil$ and $\lfloor \frac{c}{2}\rfloor$ respectively.

The framework of a multi-agent system is reviewed in the following. Consider the system that is described by the sets V and E where V contains the information of agent numbers and E the information of agent linkage. Using the terminology in the graph theory, V and E form the graph $\mathbb{G} = (V, E)$ where V and E are also known as the set of vertices or nodes, and the set of connection edges, respectively. We will thus use the terms agent, vertex, and node alternatively throughout the paper if the context is clear. A graph is useful in describing a dynamic system whose state variables change with those they are connected to. Specifically, let the nodes or vertices in V represent the state variables of some dynamic system, which evolves according to the so-called consensus equation:

$$\dot{x}_i = -\sum_{j\in\mathcal{N}_i}(x_i - x_j) \tag{1}$$

for each i in V, where $\mathcal{N}_i \subseteq V$, is the set of neighbors of agent i. The matrix form of (1) is

$$\dot{\boldsymbol{x}} = -\mathcal{L}\boldsymbol{x} \tag{2}$$

where $\boldsymbol{x} = [\,x_1\, x_2\, \ldots\, x_k\,]^T$ and k is the number of agents in the system. $\mathcal{L}$ is called the Laplacian matrix of the modeling graph and its (i, j)th entry ℓ_{ij} satisfies

$$\ell_{ij} = \begin{cases} |\mathcal{N}_i| & \text{if } i = j \\ -1 & \text{if } (i, j) \in E \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

where $|\mathcal{N}_i|$ is the number of connection edges of vertex i, or called the degree of vertex i. A graph is simple if it is unweighted, undirected, and includes no loops or multiple edges. A graph is connected if for every pair of its vertices there exists a sequence of edges connecting them. Under the standard assumptions in the literature that the modeling graph is simple and connected, the corresponding Laplacian matrix is symmetric and has an eigenvalue 0 with multiplicity 1. Let $\mathbb{G}$ be a k-vertex graph and d_i the degree of vertex i for each i in $\{1, 2, \ldots, k\}$. The conjugate d_i^* of degree d_i is defined as $|\{j|d_j \geq k-i+1\}|$, which means the number of elements in $\{j|d_j \geq k-i+1\}$. The trace $tr(\mathbb{G})$ of graph $\mathbb{G}$ is defined as the maximum integer k where there exists a nonempty vertex set $\mathcal{K} := \{i_1, i_2, \ldots, i_k\}$ such that $d_i \geq k$ for each i in $\mathcal{K}$. Suppose the vertices of $\mathbb{G}$ are numbered in the way that

$$d_1 \leq d_2 \leq \cdots \leq d_k. \tag{4}$$

If $d_{k-i} = d_{k-i}^* - 1$ for each i in $\{0, 1, \ldots, tr(\mathbb{G}) - 1\}$, then $\mathbb{G}$ is called a threshold graph. More details concerning these terms defined in the graph theory can be found in reference books (see, e.g., [17,18]).

Generally speaking, the consensus equation features a local exchange of state variable information and results in a linear autonomous system whose agents reach the common steady state. A fundamental question concerning the multi-agent system modeled by a graph is that how to design a binary control matrix $B = [\,\boldsymbol{b}_1\, \ldots\, \boldsymbol{b}_m\,]$ such that the linear time-invariant system

$$\dot{\boldsymbol{x}} = -\mathcal{L}\boldsymbol{x} + B\boldsymbol{u} \tag{5}$$

is controllable by the input $\boldsymbol{u} = [\,u_1\, \ldots\, u_m\,]^T$. Note that for $i \in \{1, 2, \ldots, m\}$, $\boldsymbol{b}_i \in \{0, 1\}^k$ is a binary control vector. k and m are the numbers of state variables and inputs respectively. The jth entry of $\boldsymbol{b}_i$ is 1 if the jth state variable directly receives the signals from the ith input, and is 0 otherwise. To simplify the expression, we will say $(-\mathcal{L}, B)$ is controllable/uncontrollable if the system in (5) is controllable/uncontrollable. There are at least two types of