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On controllability of discrete-time bilinear systems by near-controllability

Lin Tie *

School of Automation Science and Electrical Engineering, Beihang University (Beijing University of Aeronautics and Astronautics), Beijing, PR China

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ABSTRACT

In this paper, controllability of discrete-time bilinear systems is studied. By applying a recent result on near-controllability, a new sufficient condition for controllability of the systems is presented, where controllability is proved by approximation with near-controllability. The new condition is algebraically verifiable and is hence easy to apply compared with a classical result on controllability of discrete-time bilinear systems, which can be effective even when the classical result does not work. Furthermore, the control inputs to achieve the transition of the systems between any given pair of states are approximately computable according to near-controllability. Therefore, near-controllability can be used to not only better characterize the system properties, but also prove controllability with computable control inputs. The new condition is then generalized to derive similar results on controllability and near-controllability of the systems. Finally, examples are given to illustrate the results of this paper.

1. Introduction

Bilinear systems have been extensively investigated over the past decades [1–9]. Such systems are a class of nonlinear systems with a simple nonlinear structure that appears as the linear product of the control and state variable, and hence they are better understood than most other nonlinear systems [8]. Furthermore, bilinear systems can represent many phenomena and dynamics in the real world ranging from engineering to non-engineering areas, e.g. nuclear fission, biological species populations and various populations [5]. For highly nonlinear processes which cannot be modeled or approximated by the classical linear systems realistically, bilinear systems often offer relatively accurate and useful models [5–7]. It is reasonable to say that bilinear systems form a "transitional" class between the linear and the general nonlinear systems.

Controllability is clearly one important as well as fundamental issue in control theory. For bilinear systems, controllability has been considered in both continuous-time and discrete-time cases. In the continuous-time case, controllability of bilinear systems has been deeply studied profiting from the Lie algebra methods and, hence, various Lie-algebraic conditions are obtained [8]. In the discrete-time case, available results on controllability of bilinear systems are sparse, most of which are obtained under different kinds of conditions and are applicable to specific subclasses only. Even for systems in dimension two, there does not exist a general necessary and sufficient criterion for controllability. The difficulties of investigating controllability of discrete-time bilinear systems are basically due to: (i) the nonlinear nature of the systems makes the controllability problem quite complicated; and (ii) for discrete-time systems, semigroups tend to appear so that less algebraic structure of the systems is available than in the continuous-time case [10]. Indeed, controllability is a strong property which is harder to decide than accessibility, especially in the discrete-time case [10–15].

In this paper, we consider the following discrete-time bilinear system

$$x(k+1) = Ax(k) + u(k)Bx(k) = (A + u(k)B)x(k), \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state variable, $u(k) \in \mathbb{R}$ is the control, and $A, B \in \mathbb{R}^{n\times n}$ are constant matrices with $B \neq \mathbf{0}$. Many systems have natural models that are discrete bilinear or can be approximated by discrete bilinear dynamics [16,17]. Recent applications of discrete-time bilinear systems can be found in the modeling and control of complex networks [18,19]. Particularly, Boolean control networks, which have attracted a great deal of attention recently, can be converted into discrete-time bilinear systems [20].

It appears that system (1) owns a simple structure, and its controllability problem can be simply stated as well: whether there exist control inputs to steer the system from any nonzero initial state to any nonzero terminal state. If so, the system is controllable. However, from calculation we obtain that

$$x(k+1) = (A + u(k)B)(A + u(k-1)B)x(k-1)$$

* Correspondence to: Beihang University, Xueyuan Road 37, Haidian District, PR China.

E-mail address: tielin@buaa.edu.cn.

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$$
\begin{aligned}
&= \big(A^2 + u(k)BA + u(k-1)AB \\
&\quad + u(k)u(k-1)B^2\big)x(k-1) \\
&= \big(A^3 + u(k)BA^2 + u(k-1)ABA \\
&\quad + u(k-2)A^2B + u(k)u(k-1)B^2A \\
&\quad + u(k)u(k-2)BAB + u(k-1)u(k-2)AB^2 \\
&\quad + u(k)u(k-1)u(k-2)B^3\big)x(k-2) \\
&= \cdots,
\end{aligned}
$$

from which it can be seen that the control inputs are coupled not only with each other but also with the state variable in the system dynamics. The controllability problem of system (1) is essentially equivalent to the existence problem of the solution on $u(0), u(1), \ldots$ of the nonlinear equation generated by the system dynamics in a finite-time step, where the initial and terminal states are arbitrary and thus must be taken as parameters instead of certain values. However, since the system dynamics are rather complicated, it is in general quite difficult to prove whether the generated nonlinear equation has a solution for all initial and terminal states. That is, the nonlinear system dynamics make the controllability problem of system (1) difficult to deal with.

In the literature, the study of controllability of discrete-time bilinear systems was initiated from system (1) in [16,17] at the early 1970s, where necessary conditions and sufficient conditions for controllability of system (1) were provided. Particularly, to prove controllability, in [16] the matrix A was assumed to be similar to an orthogonal matrix, and in [17] the matrix B was assumed to have rank one. Later, under the same rank condition, [21] improved the work of [17] by raising necessary and sufficient controllability conditions. Since the middle 1980s, few work focusing on the controllability of system (1) has been reported until the 2000s [22–25], where [22,23] solved the controllability problem when A, B are commutative; [24] proposed some necessary conditions and sufficient conditions based on the results in [16,22]; [25] dealt with two-dimensional system (1) and gave sufficient conditions. However, the controllability problem of system (1) has not been solved and some challenges remain.

Although we are far from obtaining a necessary and sufficient condition for controllability of system (1), it is important to make progress by deriving new results even when the system is of some special or simple forms. Near-controllability is a recently established notion, which has been demonstrated on discrete-time bilinear systems to better characterize the controllability property [26–29]. For a system that is not controllable according to the definition, if we only use "uncontrollable" to describe it, we may overlook valuable properties of it. Near-controllability is thus introduced to describe those systems that are uncontrollable but have a very large controllable region, where a controllable region is such a region that, for any two states in the region, the transition of the system from one to the other can be achieved by admissible control inputs. This property was first defined and was demonstrated in [26] for a special case of system (1), i.e.

$$
x(k+1) = (I + u(k)B)x(k), \tag{2}
$$

where I is the identity matrix with appropriate dimensions. System (2) is uncontrollable if the system dimension is greater than two [22]. Nevertheless, it was shown in [26] that if B has only real eigenvalues which are nonzero and pairwise distinct, then the system's controllable region nearly covers the whole space such that the system is nearly controllable; and just recently, this result was improved in [28] by proposing a root locus approach, where a necessary and sufficient condition on near-controllability was obtained with computable control inputs. For more about near-controllability, one can refer to [27,29–31], where near-controllability of different types of discrete-time bilinear systems other than system (2) was considered and a weaker near-controllability property was studied. Note that [28] has already obtained stronger results on near-controllability of system (2) before [31], where the exact control inputs to achieve the transition of the system between any given pair of states can be easily computed by the proposed algorithm in [28]. The existing works on near-controllability are reported only for bilinear systems and the study of near-controllability is just at the beginning.

In this paper, we further demonstrate the advantage of near-controllability by using it to prove controllability. That is, we apply the near-controllability result of system (2) obtained in [28] to prove controllability of system (1) under the same condition as used in [16], i.e. A is similar to an orthogonal matrix. Specifically, the idea is to use near-controllability of system (2) to approximate controllability of system (1) through the Implicit Function Theorem [32]. A new sufficient condition on controllability of system (1) is presented. Compared with the classical result in [16] where one of the conditions is hard to verify, the presented new condition is algebraically verifiable and easy to apply, and it can be effective for checking controllability even if the classical result does not work, as shown by two examples. Also compared with the author's previous works on controllability of system (1), where [22,23] considered the case when $AB = BA$ and the system dimension has to be less than three for the system to be controllable, while the new sufficient condition does not require $AB = BA$ and works for the system with dimension greater than two; the controllability results obtained in [24] are based on the classical result in [16] and the controllability in the case of $AB = BA$, which are thus not easy to apply to system (1) with high dimensions since a similar condition like the one of the classical result needs to be verified; and [25] focused on system (1) in dimension two and presented controllability criteria which are similar to the new sufficient condition but do not require A to be similar to an orthogonal matrix.

Another contribution of this paper is that the control inputs to steer the controllable system (1) from a given initial state to a given terminal state can be approximately computed and a corresponding algorithm is proposed. Such control inputs cannot be computed if controllability is obtained by using the classical result in [16]. More specifically, for bilinear systems as well as nonlinear systems, even if controllability may have been proved, the control inputs to steer the system between an arbitrarily given pair of states are difficult to compute in most cases. Therefore, the presented new condition on controllability not only is easy to apply, but also leads to computable control inputs. Finally, extended controllability and near-controllability results for system (1) are derived based on the new condition.

This paper is organized as follows. Some preliminary results on controllability and near-controllability of discrete-time bilinear systems are given in Section 2. Section 3 is devoted to controllability of system (1), where useful lemmas for deriving the main result are also proved. Illustrative examples are provided in Section 4 and concluding remarks are made in Section 5.

2. Preliminaries and problem formulation

We first introduce the controllability definition of system (1) and then recall the classical result on controllability of system (1) in [16]. To this end, some useful definitions and notations on matrices are needed: A matrix $Q \in \mathbb{R}^{n\times n}$ is said to be *orthogonal* if it satisfies $QQ^T = I$, where $(\cdot)^T$ denotes the transpose of a matrix; A matrix $A \in \mathbb{R}^{n\times n}$ is said to be *similar to an orthogonal matrix* if there exists a nonsingular matrix $P \in \mathbb{R}^{n\times n}$ such that $Q = PAP^{-1}$, where Q is orthogonal; A matrix $C \in \mathbb{R}^{n\times n}$ is said to be *cyclic* if its characteristic polynomial is equal to its minimal polynomial, namely only one Jordan block exists for each eigenvalue in C's Jordan canonical form; $(\cdot)^{-1}$, $|\cdot|$, rank$(\cdot)$, and $\|\cdot\|_2$ respectively