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A semigroup associated to a linear control system on a Lie group

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ABSTRACT

Let us consider a linear control system Σ on a connected Lie group G. It is known that the accessibility set $\mathcal{A}$ from the identity e is in general not a semigroup. In this article we associate a new algebraic object $\mathcal{S}_\Sigma$ to Σ which turns out to be a semigroup, allowing the use of the semigroup machinery to approach Σ. In particular, we obtain some controllability results.

1. Introduction

The concept of linear control systems on Lie groups was introduced in [1] by Ayala and Tirao as the family of differential equations

$$\Sigma : \dot{g}(t) = \mathcal{X}(g(t)) + \sum_{j=1}^{m} u_j(t)\, X^j(g(t)), \tag{1}$$

where X^j are right-invariant vector fields, $u \in \mathcal{U} \subset L^\infty(\mathbb{R}, \Omega \subset \mathbb{R}^m)$ is the class of admissible controls, with $\Omega \subset \mathbb{R}^m$ a compact, convex subset satisfying $0 \in \operatorname{int} \Omega$ and the drift $\mathcal{X}$ is a linear vector field, that is, its associated flow $(\varphi_t)_{t\in\mathbb{R}}$ is a 1-parameter subgroup of the group of the automorphisms $\operatorname{Aut}(G)$ of G.

The study of linear control systems is important for at least two reasons: First, it is very well known that the classical linear system on the Euclidean space $\mathbb{R}^d$ given by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad A \in \mathbb{R}^{d\times d},\ B \in \mathbb{R}^{d\times m} \text{ and } u \in \mathcal{U}$$

is one of the most relevant control systems. Such system can be written as

$$\dot{x}(t) = Ax(t) + \sum_{j=1}^{m} u_j(t)\, b^j, \quad \text{where } b^j \in \mathbb{R}^d \text{ are the columns of } B.$$

Moreover, the flow associated with A satisfies $e^{tA} \in \mathrm{Gl}^+(d, \mathbb{R}) \subset \operatorname{Aut}(\mathbb{R}^d)$ and, since $\mathbb{R}^d$ is a commutative Lie group, any constant vector b^j is a right-invariant vector field, showing that the notion of linear systems on Lie groups is a generalization of linear systems on Euclidean spaces.

Secondly, in [2] Jouan shows that any control-affine system on a connected manifold M whose dynamic generates a finite Lie algebra is diffeomorphic to a linear control system on a Lie group or on a homogeneous space, showing that the understanding of the behavior of the system Σ is in fact very important in applications.

One of the ways to analyze the dynamical behavior of control systems on Lie groups or homogeneous spaces is via semigroup theory by relating the solutions of the system with the action of a semigroup. For instance, for right-invariant systems the reachable set from the identity of the group is a semigroup and so all the machinery of the semigroup theory is available (see [3]). Different from right-invariant systems, the reachable set of a linear control system on a nonabelian Lie group cannot be a semigroup unless it is the whole group (see [4], Section 4). In this paper we show that, despite this fact, the reachable set strictly contains a semigroup that is intrinsically connected with the controllability properties of the system.

The paper is structured as follows: In Section 2 we introduce some G-subgroups associated to a given linear vector field. We define the reachable set of a linear system and show the relation between this set with the subgroups induced by its drift. In Section 3 we define the semigroup associated to a given linear system. In order to assure controllability of the system we show that it is enough to analyze the semigroup. Moreover, if we assume that the reachable set is open our work reduces the problem from an arbitrary Lie group G to a nilpotent G-subgroup. At the end of the section we show that for semisimple Lie groups the mentioned semigroup has nonempty interior if and only if it is the

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whole group if and only if the associated linear control system is controllable.

2. Preliminaries

In this section we give the background needed about linear control systems and group decomposition induced by linear vector field.

Let us assume from here on that G is a connected Lie group with Lie algebra $\mathfrak{g}$ and dimension d, where $\mathfrak{g}$ is identified with the set of the right-invariant vector fields. For any linear vector field $\mathcal{X}$ on G let $\mathcal{D}$ be the $\mathfrak{g}$-derivation determined by $\mathcal{X}$ (see for instance [5]). The relation between $\mathcal{D}$ and the flow $(\varphi_t)_{t\in\mathbb{R}}$ of $\mathcal{X}$ is given by

$$(d\varphi_t)_e = \mathrm{e}^{t\mathcal{D}}, \quad t \in \mathbb{R}.$$

Consider the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$ and the derivation $\mathcal{D}_{\mathbb{C}}$ on $\mathfrak{g}_{\mathbb{C}}$ induced by $\mathcal{D}$. For an eigenvalue α of $\mathcal{D}$ we defined the α-generalized eigenspace of $\mathcal{D}_{\mathbb{C}}$ as

$$(\mathfrak{g}_{\mathbb{C}})_\alpha = \{X \in \mathfrak{g}_{\mathbb{C}} : (\mathcal{D}_{\mathbb{C}} - \alpha)^n X = 0 \text{ for some } n \geq 1\}.$$

Since the eigenvalues of $\mathcal{D}_{\mathbb{C}}$ coincide with the ones of $\mathcal{D}$ we have that

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_\alpha (\mathfrak{g}_{\mathbb{C}})_\alpha, \quad \text{for every } \alpha \text{ eigenvalue of } \mathcal{D}.$$

Moreover, if β is also an eigenvalue of $\mathcal{D}$ Proposition 3.1 of [6] implies

$$[(\mathfrak{g}_{\mathbb{C}})_\alpha, (\mathfrak{g}_{\mathbb{C}})_\beta] \subset (\mathfrak{g}_{\mathbb{C}})_{\alpha+\beta}, \tag{2}$$

where $(\mathfrak{g}_{\mathbb{C}})_{\alpha+\beta} = \{0\}$ when $\alpha + \beta$ in not an eigenvalue of $\mathcal{D}$.

Let

$$\mathfrak{g}_{\mathbb{C}}^+ = \bigoplus_{\alpha : \mathrm{Re}(\alpha)>0} (\mathfrak{g}_{\mathbb{C}})_\alpha, \quad \mathfrak{g}_{\mathbb{C}}^0 = \bigoplus_{\alpha : \mathrm{Re}(\alpha)=0} (\mathfrak{g}_{\mathbb{C}})_\alpha \quad \text{and}$$

$$\mathfrak{g}_{\mathbb{C}}^- = \bigoplus_{\alpha : \mathrm{Re}(\alpha)<0} (\mathfrak{g}_{\mathbb{C}})_\alpha.$$

Since $\mathfrak{g}_{\mathbb{C}}^+$, $\mathfrak{g}_{\mathbb{C}}^0$ and $\mathfrak{g}_{\mathbb{C}}^-$ are invariant by conjugation they coincide with the complexification of

$$\mathfrak{g}^+ := \mathfrak{g}_{\mathbb{C}}^+ \cap \mathfrak{g}, \qquad \mathfrak{g}^0 := \mathfrak{g}_{\mathbb{C}}^0 \cap \mathfrak{g} \quad \text{and} \quad \mathfrak{g}^- := \mathfrak{g}_{\mathbb{C}}^- \cap \mathfrak{g}.$$

Moreover, $\mathfrak{g}^+$, $\mathfrak{g}^0$ and $\mathfrak{g}^-$ are $\mathcal{D}$-invariant Lie subalgebras of $\mathfrak{g}$, with $\mathfrak{g}^+$ and $\mathfrak{g}^-$ nilpotent ones, and such that $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^-$. Also, $\mathfrak{g}^+$ and $\mathfrak{g}^-$ are ideals of the Lie subalgebras $\mathfrak{g}^{+,0} := \mathfrak{g}^+ \oplus \mathfrak{g}^0$ and $\mathfrak{g}^{-,0} := \mathfrak{g}^- \oplus \mathfrak{g}^0$, respectively. Let us denote by, $G^+, G^0, G^-, G^{+,0}$ and $G^{-,0}$ the connected Lie subgroups of G with Lie algebras $\mathfrak{g}^+$, $\mathfrak{g}^0$, $\mathfrak{g}^-$, $\mathfrak{g}^{+,0}$ and $\mathfrak{g}^{-,0}$ respectively.

By Proposition 2.9 of [7] the subgroups $G^+, G^0, G^-, G^{+,0}$ and $G^{-,0}$ are closed subgroups of G that are invariant by the flow $(\varphi_t)_{t\in\mathbb{R}}$.

From here, we focus in a linear control system Σ on a connected Lie group G as follows

$$\Sigma : \dot{g}(t) = \mathcal{X}(g(t)) + \sum_{j=1}^m u_j(t) X^j(g(t)),$$

where $\mathcal{X}$ is a linear vector field, X^j are right-invariant vector fields and $u \in \mathcal{U}$. For any $g \in G$, $u \in \mathcal{U}$ and $\tau \in \mathbb{R}$ we denote by $\phi_{\tau,u}(g)$ the solution of Σ at time τ with initial condition g and control u. If $\tau > 0$

$$\mathcal{A}_\tau(g) := \{\phi_{\tau,u}(g) : u \in \mathcal{U}\} \quad \text{and} \quad \mathcal{A}(g) := \bigcup_{\tau>0} \mathcal{A}_\tau(g), \tag{3}$$

are the *reachable set from g at time τ* and the *reachable set of g*, respectively. We also denote $\mathcal{A}_\tau(e) = \mathcal{A}_\tau$ and $\mathcal{A}(e) = \mathcal{A}$. We will say that the system Σ satisfies the *rank*-condition if

$$\mathrm{span}_{\mathcal{L}}\{\mathcal{D}^i(X^j), i \in \mathbb{N}_0, j = 1, \ldots, m\} = \mathfrak{g}, \quad \text{where} \quad \mathbb{N}_0 := \mathbb{N} \cup \{0\},$$

that is, if the smallest Lie subalgebra of $\mathfrak{g}$ that is $\mathcal{D}$-invariant coincides with $\mathfrak{g}$.

The next proposition states the main properties of the reachable sets. Its proof can be found in [5] Proposition 2.

Proposition 2.1. *It holds:*

1. if $0 \leq \tau_1 \leq \tau_2$ then $\mathcal{A}_{\tau_1} \subset \mathcal{A}_{\tau_2}$
2. for all $g \in G$ we have $\mathcal{A}_\tau(g) = \mathcal{A}_\tau \varphi_\tau(g)$
3. for all $\tau_2, \tau_1 \geq 0$ we have $\mathcal{A}_{\tau_1+\tau_2} = \mathcal{A}_{\tau_1}\varphi_{\tau_1}(\mathcal{A}_{\tau_2}) = \mathcal{A}_{\tau_2}\varphi_{\tau_2}(\mathcal{A}_{\tau_1})$.

Definition 2.2. The system will be said to be controllable if $\mathcal{A} = G$.

In [8] the authors study the controllability property of linear control systems and introduce the following notion.

Definition 2.3. Let G be a connected Lie group. We say that the Lie group G has finite semisimple center if each semisimple Lie subgroup of G has finite center.

Remark 2.4. For instance, a connected Lie group G has finite semisimple center if G has one (and hence all) Levi subgroup L with finite center. Actually, any semisimple Lie subgroup of G is conjugated to a subgroup of L. These facts come from Malcev's Theorem (see [9] Theorem 4.3) and its corollaries.

Assume that G has finite semisimple center. The relation between the subgroups induced by $\mathcal{X}$ and the linear control system Σ is given by the next result (see Theorem 3.9 of [8]).

Theorem 2.5. *If $\mathcal{A}$ is open then $G^{+,0} \subset \mathcal{A}$.*

Remark 2.6. By Theorem 3.5 of [1], one way to assure the openness condition on $\mathcal{A}$ is by the *ad-rank* condition. The system Σ is said to satisfy the ad-rank condition if the vector space

$$\mathrm{span}_{\mathcal{V}}\{\mathcal{D}^i(X^j), i \in \mathbb{N}_0, j = 1, \ldots, m\},$$

coincides with $\mathfrak{g}$. When Σ satisfies the ad-rank condition we have that $e \in \mathrm{int}\, \mathcal{A}_\tau$ for all $\tau > 0$ which implies, in particular, that $\mathcal{A}$ is an open subset.

3. Semigroup associated to a linear control system on a Lie group

Since the positive orbit $\mathcal{A}$ of a linear control system Σ is in general not a semigroup (see for instance [4], Section 4), in this section we associate to Σ a new algebraic object $\mathcal{S}_\Sigma$ which turns out to be a semigroup. In particular, $\mathcal{S}_\Sigma$ enables us to pass from the control theory of linear systems to the theory of semigroups. Furthermore, controllability of Σ is equivalent to $\mathcal{S}_\Sigma = G$.

From here we assume that the linear system Σ on G satisfies the rank-condition. As before, we denote by $(\varphi_t)_{t\in\mathbb{R}}$ the 1-parameter group of automorphisms associated to the drift $\mathcal{X}$ of Σ. Next, we introduce the subset $\mathcal{S}_\Sigma \subset G$ defined by

$$\mathcal{S}_\Sigma := \bigcap_{t\in\mathbb{R}} \varphi_t(\mathcal{A}).$$

Since $\varphi_t(e) = e$ for all $t \in \mathbb{R}$ and $e \in \mathcal{A}$ it follows that $\mathcal{S}_\Sigma \neq \emptyset$.

Proposition 3.1. *With the previous notations it holds*

1. *$\mathcal{S}_\Sigma$ is the greatest φ-invariant subset of $\mathcal{A}$*
2. *For any $\tau_0 \geq 0$*

$$\mathcal{S}_\Sigma = \bigcap_{t\geq\tau_0} \varphi_t(\mathcal{A})$$