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A moment-based approach to ensemble controllability of linear systems

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ABSTRACT

In this paper we introduce a new perspective on the L^2-ensemble controllability problem of linear time-invariant systems using an ℓ^2-framework. The reformulation results from focussing on the controllability of the ensemble's moments, which evolve under a linear system defined on the space of square-summable sequences. For a specific class of ensembles, a necessary and sufficient condition can be stated in terms of truncations of infinite Kalman matrices. We illustrate the analysis in this framework on several examples which highlight the possibility of using elementary structural arguments in proving or disproving the controllability of an ensemble's moments, as well as essential differences to the uniform ensemble controllability property which is mostly considered in the literature.

1. Introduction

Practical problems in a diversity of different applied fields dealing with populations of systems gave rise to the recent study of their fundamentals in control theory [1–4]. Under the name of ensemble control, the systems theoretic basics of these problems dealing with populations of systems, such as in cell biology [5–8], nuclear magnetic resonance imaging [9] or process control [10], are studied in a control theoretic framework. The common theme in these problems centering around populations of systems is the manipulation/control of an ensemble of nearly identical dynamical systems with the severe restriction of having only one input to control the whole family of systems. While this sounds like a difficult task from an intuitive point of view, certain parameter variations in the dynamics of the individual systems can allow one to gain controllability over the ensemble. The precise systems theoretic study of controllability of ensembles is the subject of ensemble control. In the following we give a brief review of the state of the art of ensemble control.

1.1. Ensemble control of linear parameter-dependent systems

A typical model studied in the ensemble control of linear systems is given by

$$\frac{\partial}{\partial t}x(t,\theta) = A(\theta)x(t,\theta) + B(\theta)u(t), \quad x(0,\theta) = x_0(\theta), \tag{1}$$

where θ is a parameter that typically varies in a compact interval $[a, b] \subset \mathbb{R}$ and $x_0(\cdot)$ is continuous, see e.g. [4] and references therein. This system models ensembles of linear systems with the same dynamical structure but heterogeneous parameters θ. Practical limitations in typical ensemble control problems furthermore require that an ensemble can only be controlled by applying a common control input as opposed to applying individual input signals to individual systems, which is the crucial difficulty in the control of ensembles.

A typical control task is to steer the ensemble to a desired terminal state at time $t = T$, which is often desired to be able to vary with the specific parameter θ. Therefore, terminal states are specified as functions $\theta \mapsto x(T, \theta)$. Again, the control input on the other hand is required to be independent of the parameter θ. Due to the infinite-dimensional nature of this problem, exact ensemble controllability cannot be expected so that one has to accept that one can only steer the ensemble's state to a desired terminal state within a given accuracy which, however, can be made arbitrarily small. There are different ways in which the mismatch between the ensemble's terminal state and the desired terminal state can be measured, such as in terms of an L^p-error given by

$$\|x(T,\cdot) - x^\star(\cdot)\|_{L^p([a,b],\mathbb{R}^n)} = \left(\int_a^b \|x(T,\theta) - x^\star(\theta)\|^p \, d\theta\right)^{\frac{1}{p}}.$$

A first theoretical study of the L^2-*ensemble controllability* problem for ensembles of linear time-varying systems led to conditions that

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are stated in terms of a singular value decomposition of the input-to-state operators [3]. The main drawbacks of such an approach are the need for the singular value composition, which is slightly impractical, and the fact that the connection to the original system (1) is missing in the aforementioned conditions. Therefore, starting with [4,11], most of the attention in the ensemble control of linear systems has shifted towards the problem of *uniform ensemble controllability* in which the error is measured in the uniform norm, i.e.

$$\|x(T,\cdot)-x^{\star}(\cdot)\|_{\infty}=\sup_{\theta\in[a,b]}\|x(T,\theta)-x^{\star}(\theta)\|.$$

In this framework, the problem of ensemble controllability is translated into a polynomial approximation problem and more practical results can be formulated. Since uniform ensemble controllability constitutes a stronger notion than L^2-ensemble controllability, the results obtained in the uniform ensemble controllability problem also apply to L^2-ensemble controllability [4].

1.2. Uniform ensemble controllability through polynomial approximation

A constructive approach to the study of uniform ensemble controllability of linear systems was introduced in [4,11] and also pursued by [12]. The key idea is best illustrated via discrete-time linear systems. Consider the ensemble

$$x(k+1,\theta)=A(\theta)x(k,\theta)+B(\theta)u(k),$$

where $\theta \in [a,b] \subset \mathbb{R}$. If we assume that $x(0,\theta) = 0$ for all $\theta \in [a,b]$, then the evolution of the parameterized ensemble is given by

$$x(k,\theta)=\sum_{p=0}^{k-1}A(\theta)^pB(\theta)u(k-p).$$

At a given time step k, the right-hand side is a polynomial in $A(\theta)$, and the inputs $u(k)$ can be seen as coefficients of the polynomial. Thus, the natural idea is to view the ensemble control problem as a polynomial approximation problem. Given a continuous function $\theta \mapsto x_F(\theta)$ as the desired terminal state, one studies whether a given ensemble $(A(\theta), B(\theta))$ can generate a polynomial which approximates the desired terminal state arbitrarily precise in the uniform norm. This question is of course closely related to the study of the span of the polynomials $A(\theta)^pb_j(\theta)$ in both the discrete time and the continuous time cases. This approach has been pursued by [4,11] and more recently also by [12] to obtain more constructive results than those L^p-ensemble controllability results obtained through earlier operator theoretic approaches, which study the range space of the reachability operator $\mathcal{R}_T : C([a,b],\mathbb{R}^n)\to C([a,b],\mathbb{R}^n)$ defined by

$$(\mathcal{R}_Tx_0)(\theta)=e^{A(\theta)T}x_0(\theta)+\int_0^T e^{A(\theta)(T-\tau)}B(\theta)u(\tau)\,d\tau=x(T,\theta).$$

However, the analysis in the ensemble control of linear systems is still rather elaborate in this polynomial approximation framework. Therefore, in this work we propose a closely related but quite different "moment-based approach". This approach allows for a different line of attack that may be considered conceptually simpler and leads to novel viewpoints of the ensemble control problem.

2. Ensemble control through control of moments

The approach using polynomials to approximate the terminal state presented in the last subsection motivated us to consider the inherently related idea of "moment-based approaches". This approach, in particular, leads to a natural way to analyze L^2-ensemble controllability.

2.1. Moments of scalar functions

First of all, recall that any continuous scalar-valued function $f : [a,b] \to \mathbb{R}$ is uniquely determined by the integral values

$$\xi^{(p)}:=\int_a^b x^pf(x)\,dx,$$

for $p \in \mathbb{N}_0$. This is because any such function f is uniquely determined by its values when integrated against continuous functions, and moreover, since any continuous function on a compact interval can be approximated by polynomials to arbitrary accuracy. Even though the approach presented in this section is closely related to the ideas pursued in [4,11,12], this will eventually result in a quite different framework.

Given the basic viewpoint introduced above, the idea is to consider the control of the *moments* of the ensemble, defined by

$$\xi^{(p)}(t):=\int_a^b \theta^px(t,\theta)\,d\theta,$$

where $\xi^{(p)}(t)$ can be thought of as the vector of moments of the components $x_i(t,\theta)$. Since for each $i \in \{1,\dots,n\}$ the moments $\xi_i^{(p)}(t)$, where $p \in \mathbb{N}_0$, determine the ith component $x_i(t,\theta)$ uniquely, $\xi^{(p)}$ for $p \in \mathbb{N}_0$ determine the function $\theta \mapsto x(t,\theta)$ uniquely. This key idea reduces the ensemble controllability problem to the question of whether or not we can assign $\xi^{(1)}(t), \xi^{(2)}(t), \dots$ an arbitrary value at time $t = T$, or, which is to be expected in view of results in [4,11,12], *approximately* assign a value to $\xi^{(1)}(t), \xi^{(2)}(t), \dots$. This is referred to as moment-controllability of an ensemble in this paper.

We stress that while most approaches in infinite-dimensional systems theory rather focus on reducing such a countably infinite-dimensional and *spatially invariant* problem to a closed, but uncountably infinite-dimensional description (see e.g. [13]), in this paper, we work exactly in the opposite direction. As we will see, this slightly more "unconventional" approach yields an interesting novel viewpoint and is very well able to produce constructive results.

2.2. Moment dynamics of a specific class of ensembles

To briefly illustrate what is to be expected of this approach, we consider an ensemble

$$\frac{\partial}{\partial t}x(t,\theta)=\theta Ax(t,\theta)+Bu(t),$$

i.e. ensembles where the control vector field is independent of θ and where one can pull out the parameter θ from the system matrix, see e.g. [12]. Differentiating the pth order moment $\xi^{(p)}$, we see that

$$\frac{d}{dt}\xi^{(p)}(t)=\frac{d}{dt}\int_a^b\theta^px(t,\theta)\,d\theta=\int_a^b\theta^{p+1}Ax(t,\theta)+\theta^pBu(t)\,d\theta$$
$$=A\xi^{(p+1)}(t)+\left(\int_a^b\theta^p\,d\theta\right)Bu(t),$$