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Parameterization of positively stabilizing feedbacks for single-input positive systems[☆]

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ABSTRACT

This paper provides parameterizations of all positively stabilizing feedbacks for a particular class of finite-dimensional single-input positive systems, a method to yield these parameterizations by means of tools of linear programming and a convergence analysis which allows to extend the results to a particular infinite-dimensional system described by a parabolic partial differential equation. It also provides an academic standard example – the pure diffusion equation – to support the theoretical results.

1. Introduction

Positive linear systems are linear systems whose state variables are nonnegative at all time. Studying this kind of systems is of great importance as the nonnegativity property can be found frequently in numerous fields like biology, chemistry, physics, ecology, economy or sociology (see e.g. [1–6] for particular examples). When stabilizing a system, one cannot force a state variable representing e.g. a mass, a density or a concentration to become negative at some time in order to make it asymptotically stable. It is then essential to force the nonnegativity of the state at all time when studying positive systems to ensure that the very nature of the system is preserved and that the mathematical methodology makes sense from the point of view of the applications.

In this paper we deal with the positive stabilization of positive systems. The main contributions include equivalent parameterizations of all positively stabilizing feedbacks for a particular class of positive systems, using linear programming tools [7] among other things, an in-depth analysis of the pure diffusion system in which the theoretical results are applied, and a convergence study of the discretized system to the nominal one—showing the consistency and stability [8,9] of the numerical scheme and using the state space approach [10].

This paper is organized as follows. In Section 2 we provide the reader with the definitions, the main concepts and the notations used in the paper. We give in Section 3 a method to yield all the positively stabilizing feedbacks for a particular class of finite-dimensional single-input positive systems and apply the results to a classical example, namely the pure diffusion system, by means of the finite difference method. In Section 4 we show the convergence of the discretized system to the nominal one, thus leading to a positively stabilizing feedback boundary control for the partial differential equation (PDE) system. We validate the finite-dimensional results in Section 5 by providing the reader with numerical simulations. Finally, Section 6 brings a conclusion, together with some perspectives.

2. Preliminaries

2.1. Terminology

In the sequel, we will use the sets $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x \geq 0\}$, $\mathbb{R}_{0,+} := \{x \in \mathbb{R} \mid x > 0\}$, $\mathbb{R}_+^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \in \mathbb{R}_+, \forall i = 1, \dots, n\}$ and $\mathbb{R}_{0,+}^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \in \mathbb{R}_{0,+}, \forall i = 1, \dots, n\}$. Similarly, $\mathbb{R}_-$, $\mathbb{R}_{0,-}$, $\mathbb{R}_-^n$ and $\mathbb{R}_{0,-}^n$ denote the sets $\{x \in \mathbb{R} \mid x \leq 0\}$, $\{x \in \mathbb{R} \mid x < 0\}$, $\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \in \mathbb{R}_-, \forall i = 1, \dots, n\}$ and $\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \in \mathbb{R}_{0,-}, \forall i = 1, \dots, n\}$ respectively. The real part of a complex number $z \in \mathbb{C}$ will be denoted by $\mathcal{R}(z)$. A *nonnegative* vector v has all its components

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greater or equal to zero (i.e. $v_i \in \mathbb{R}_+$, for all i). The transpose of a matrix A will be denoted by A^T. The rank of a matrix A will be denoted by $rk(A)$. The ijth entry of a matrix A will be denoted by a_{ij}. The spectrum of a matrix A is the set of its eigenvalues and will be denoted by $\sigma(A)$. A *nonnegative* matrix A (denoted by $A \geq 0$) has all its entries greater or equal to zero (i.e. $a_{ij} \in \mathbb{R}_+$, for all i, j). A *Metzler* matrix A has all its off-diagonal entries greater or equal to zero (i.e. $a_{ij} \in \mathbb{R}_+$, for all $i \neq j$). A *stable* matrix A has all its eigenvalues with negative real parts (i.e. $\mathcal{R}(\lambda) < 0$, $\forall \lambda \in \sigma(A)$). A cone C is *polyhedral* if $C = \{x \mid Ax \leq 0\}$ for some matrix A. The cone *(finitely) generated by* the vectors $x_1, \ldots, x_n$ is the set $cone\{x_1, \ldots, x_n\} := \{\alpha_1 x_1 + \cdots + \alpha_n x_n \mid \alpha_1, \ldots, \alpha_n \geq 0\}$, i.e. the smallest convex cone containing $x_1, \ldots, x_n$. The *interior* of the cone generated by the vectors $x_1, \ldots, x_n$ will be denoted by $cone^o\{x_1, \ldots, x_n\}$. For convenience, lower-case letters when used in an appropriate context will represent scalars or vectors, while upper-case letters will represent matrices.

2.2. Main concepts

Let a linear time-invariant system

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times m}$. We first recall the concept of positive linear system [4,5,11,12].

Definition 1. A linear system $R = [A, B, C, D]$ is positive if for every nonnegative initial state $x_0 \in \mathbb{R}^n_+$ and for every admissible nonnegative input u (i.e. every piecewise continuous function $u : \mathbb{R}_+ \to \mathbb{R}^m_+$) the state trajectory x of the system and the output trajectory y are nonnegative (i.e. for all $t \geq 0$, $x(t) \in \mathbb{R}^n_+$ and $y(t) \in \mathbb{R}^p_+$).

It is possible to express the positivity of a system by use of the matrices A, B, C and D only [4,5].

Theorem 1. *A linear system $R = [A, B, C, D]$ is positive if and only if A is a Metzler matrix and B, C and D are nonnegative matrices.*

Now we define the positive stabilizability of positive systems. For convenience, throughout the paper the notion of stability will refer to asymptotic stability, which is equivalent to exponential stability as we deal with LTI systems.

Definition 2. A positive linear system $R = [A, B, C, D]$ is positively (exponentially) stabilizable if there exists a state feedback matrix $K \in \mathbb{R}^{m\times n}$ such that $A + BK$ is a stable Metzler matrix, i.e. such that there exist positive constants M and σ such that for all $t \geq 0$

$$\left\| e^{(A+BK)t} \right\| \leq Me^{-\sigma t}$$

and for all $t \geq 0$, $e^{(A+BK)t} \geq 0$. Such a feedback matrix K is called a positively stabilizing feedback for the system R.

The positive stabilization problem is concerned with existence conditions and the computation of such a matrix K. Finally, we introduce an important result from [2,5,12] which provides a necessary and sufficient condition for the stability of a Metzler matrix.

Lemma 1. *A Metzler matrix $A \in \mathbb{R}^{n\times n}$ is stable if and only if there exists v in $\mathbb{R}^n_{0,+}$ such that Av is in $\mathbb{R}^n_{0,-}$.*

Remark 1. In Lemma 1 we use the notation $v \in \mathbb{R}^n_{0,+}$ for a strictly positive vector instead of the notation $v \gg 0$ which can often be found in the literature.

Remark 2. The sufficiency of the condition can be shown by considering the Lyapunov function $V(x) = v^T x$ which leads to $\dot{V}(x) = v^T Ax < 0$. The necessity follows from the fact that the opposite of the inverse of a stable Metzler matrix is nonnegative: it suffices to define $v = -A^{-1}\tau$ with $\tau \in \mathbb{R}^n_{0,+}$. See [5, Lemma 2.2] or [13, Lemma 1.1].

3. Designing all positively stabilizing feedbacks

3.1. A particular class of systems

Let us consider a particular class of systems and provide the reader with a systematic way to design the general expression of any positively stabilizing feedback. More precisely, we consider single-input LTI positive systems described by $\dot{x} = Ax + bu$ where $A \in \mathbb{R}^{n\times n}$ is Metzler and $b \in \mathbb{R}^n$ is nonnegative and has only one non-null entry (w.l.o.g. the first one). The particular structure of b is pretty common, notably when applying finite differences to PDE systems with boundary control (see Section 3.2). The following result is analogous to [14, Theorem 3.1], though the first concerns single-input multi-output systems with state-feedback and the second concerns multi-input single-output systems with output-feedback.

Theorem 2. *Consider a linear time-invariant positive system $R = [A, B, C, D]$ described by $\dot{x} = Ax + bu$ where A is Metzler and b is nonnegative and has only its first entry different from zero. Then*
(a) $k = [k_1 \;\; \ldots \;\; k_n]$ is a positively stabilizing feedback for R if and only if

$$k_1 = \frac{-a_{11}v_1 - (a_{12} + b_1 k_2)v_2 - \cdots - (a_{1n} + b_1 k_n)v_n - \omega}{b_1 v_1}$$

and

$$k_i \geq \frac{-a_{1i}}{b_1} \quad i = 2, \ldots, n,$$

where $\omega > 0$ is a free parameter, and $v \in \mathbb{R}^n$ is positive and solution of the strict inequalities set

$$-a_{i1}v_1 - \cdots - a_{in}v_n > 0 \qquad i = 2, \ldots, n. \tag{1}$$

(b) The set of solutions of (1) with the positivity constraint over v is given by $\overset{o}{cone}\{s_1, \ldots, s_r\}$ where $r \leq 2n - 1$ and the column vectors $s_1, \ldots, s_r$ are such that $cone\{a_2^T, \ldots, a_n^T, -e_1, \ldots, -e_n\} = \{x \mid s_1^T x \leq 0, \ldots, s_r^T x \leq 0\}$ where a_i denotes the ith row of A and e_i the ith vector of the canonical basis of $\mathbb{R}^n$.

Remark 3. For the pure diffusion system, one can show that $r = n$ (see Theorem 3 in Section 3.2).

Proof. (a) It is straightforward to show that any feedback k as described above positively stabilizes the system. Now consider a general feedback $k = [k_1 \;\; \ldots \;\; k_n]$. The closed-loop matrix $A + bk$ has to be Metzler as we want positivity to be maintained, which yields the conditions

$$k_i \geq \frac{-a_{1i}}{b_1} \qquad i = 2, \ldots, n$$

with k_1 free. For the stability property we use Lemma 1: $A + bk$ is stable if and only if one can find a vector $v \in \mathbb{R}^n_{0,+}$ such that $(A + bk)v$ is in $\mathbb{R}^n_{0,-}$. This leads to the following set of strict inequalities

$$\begin{aligned} -(a_{11} + b_1 k_1)v_1 - \cdots - (a_{1n} + b_1 k_n)v_n &= \omega \\ -a_{i1}v_1 - \cdots - a_{in}v_n &> 0 \qquad i = 2, \ldots, n \\ v_i &> 0 \qquad i = 1, \ldots, n \end{aligned}$$

with $\omega > 0$. As only the first equation depends on the entries of k, we can express

$$k_1 = \frac{-a_{11}v_1 - (a_{12} + b_1 k_2)v_2 - \cdots - (a_{1n} + b_1 k_n)v_n - \omega}{b_1 v_1}$$

with v any solution of the strict inequalities set above.