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Zero-dynamics design and its application to the stabilization of implicit systems

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ABSTRACT

We present a formula that computes the output of an R-controllable, regular, single-input linear time-invariant implicit system in such a way that it has prescribed relative degree and zeros. The formula is inspired on different generalizations of Ackermann's formula.

A possible application is in the context of sliding-mode control of implicit systems where, as the first step, one can use the proposed formula to design a sliding surface with desired dynamic characteristics and, as the second step, apply a higher-order sliding-mode controller to enforce a sliding motion along the resulting sliding surface.

1. Introduction

In order to derive a mathematical model of a given dynamical system one chooses first a set of descriptor variables (position, speed, acceleration, temperature, current, voltage etc.) in an attempt to define the state. The relationship among the chosen variables gives rise to differential or algebraic equations, sometimes resulting in an implicit system. Implicit systems are also referred to as generalized, descriptor, differential–algebraic (DAE) or semi-state systems, and are mainly motivated by applications in electric circuits and electromechanical or mechanical systems such as constrained robots.

It is possible to bring a single-input–single-output explicit system with strictly positive relative degree into a normal form that clearly reveals its zero dynamics. If the system is minimum phase, that is, if the zero dynamics are stable, it is then possible to stabilize the system by means of a simple state feedback (it suffices to drive the system output to zero). The extension of such results to the case of implicit systems was reported, e.g., in [1–3], where the authors propose a normal form for implicit systems and analyze the stability of its zero dynamics.

The problem of choosing an output with desired zeros is referred to as *zero placement* [4]. Since the zeros of the transfer function of any linear time-invariant (LTI) system, explicit or implicit, coincide with the eigenvalues of its zero dynamics, the problem of zero placement can be assimilated to the problem of defining the eigenvalues of the zero dynamics.

There are several circumstances in which one might be interested in designing an output that induces specific zero dynamics. In sliding-mode control (SMC), for example, the strategy consists in two steps: the design of a so-called sliding surface and the design of the actual control law, whose goal is to bring the system state onto the sliding surface and constrain the state to slide along it thereafter [5]. In the SMC literature, the system behavior when sliding along the sliding surface is called the sliding dynamics. A closer look at the methodology reveals that the sliding dynamics are nothing else than the zero dynamics of a virtual output, called the sliding variable. A usual recipe to the design of the sliding surface is the application of a formula by Ackermann and Utkin [6]. The two-step approach results in a controlled system which is completely insensitive to a large class of external disturbances. From an application point of view, this robustness presents an advantage over the simpler strategy consisting on the application of Ackermann's formula directly (i.e., as opposed to Ackermann–Utkin's formula) in order to specify the eigenvalues of the dynamics on the entire state-space (i.e., as opposed to the lower-dimensional sliding surface).

The original formula by Ackermann and Utkin is restricted to sliding surfaces of co-dimension one, which implies that the sliding variable has relative degree one. This is natural in the context of conventional SMC, since step two requires the sliding variable to have relative degree precisely equal to one. However, modern (higher-order) SMC removes the restriction on the relative degree of the sliding surface in step two. It is then reasonable to adjust step one and aim at sliding surfaces with desired sliding dynamics and of co-dimension higher than one. This motivates the generalization of the formula by Ackermann and Utkin presented in [7]. The

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objective of this paper is to further extend the formula to the case of regular LTI implicit systems.

A formula to design a stabilizing state feedback for completely controllable (C-controllable) implicit systems, based on Ackermann's formula for explicit systems, can be found in [8]. Such formula does not require the implicit system to be in the so-called Weierstrass or quasi-Weierstrass form. Obviating the need to use Weierstrass' form, which can be thought of as a generalization of Jordan's form, represents an advantage in practical terms, since similarity transformations can sometimes induce large errors in the presence of parameter uncertainties [9]. The formula proposed here is more general, as it works for R-controllable systems (R-controllability is weaker than C-controllability) and serves to specify the zero dynamics instead of the system dynamics in the complete state space.

Other than the higher-order SMC application mentioned above, the main result can also be used to design an output such that the system is minimum phase and has relative degree one or zero. The closed loop is thus feedback equivalent to a passive system and any passivity-based techniques can be used to control it.

The paper structure is as follows: In Section 2 we introduce the basic theory for singular systems and state the problem formally. The main result is presented in Section 3. In Section 4 we analyze the implications of our main result in the stabilization problem of implicit systems and present a concrete example. Conclusions and future work are presented in Section 5.

2. Preliminaries

Consider the single-input LTI implicit system

$$E\dot{x} = Ax + Bu \tag{1a}$$

$$y = Cx, \tag{1b}$$

where $x \in \mathbb{R}^n$ and $u, y \in \mathbb{R}$ are the state, the control input and the output at time t, respectively (we omit the time arguments to ease the notation). The matrices $E, A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times 1}$ are constant and given. We have rank $E = n_0 < n$. The output matrix $C \in \mathbb{R}^{1\times n}$, not given *a priori*, will be specified later.

Definition 1 (*[10]*). For any two matrices $E, A \in \mathbb{R}^{n\times n}$, the pencil $\lambda E - A$ is called *regular* if the determinant $\det(\lambda E - A)$ does not vanish identically.

If $\det(\lambda E - A) \equiv 0$ or if the matrices are non square, then the pencil is called singular [10]. System (1a) is called solvable if, for any admissible input and any given admissible initial condition, Eq. (1a) has a unique solution [11]. This happens when the pencil $\lambda E - A$ is regular [10]. In such a case, the implicit system (1a) is called a regular implicit system.

An implicit system is regular if, and only if, there exist nonsingular matrices L and R such that, by applying the coordinate transformation

$$\begin{bmatrix} x_s \\ x_f \end{bmatrix} = R^{-1}x, \quad x_s \in \mathbb{R}^{n_1}, \quad x_f \in \mathbb{R}^{n_2},$$

and multiplying (1a) on the left by L we obtain

$$\dot{x}_s = A_s x_s + B_s u \tag{2a}$$

$$N\dot{x}_f = x_f + B_f u, \tag{2b}$$

where N is nilpotent with index of nilpotence q (see [12] for details). System (1a) is called an implicit system with index q, for short. If the matrices A_s and N are in Jordan form, then system (2) is said to be in Weierstrass form [10], otherwise, system (2) is said to be in quasi-Weierstrass form [13]. Recall that $\deg\det(\lambda E - A) = n_1 < n$, where the function deg represents the degree of a polynomial [12]. The set of finite eigenvalues of a matrix pair (E, A) is denoted as $\Lambda(E, A) = \{\lambda_1, \lambda_2, \ldots, \lambda_{n_1}\}$.

The solution of subsystem (2a) can be easily determined from well-known results on explicit systems [14]. The solution of Subsystem (2b) depends affinely on u and its first $q-1$ time derivatives [12,15,11]. Let U be the set of admissible input functions. In order to assure the continuity of x_f we require $j = \max\{i \in \mathbb{N} : \operatorname{Im} B_f \not\subseteq \ker N^i\}$ and $U = \mathcal{C}^j$, where $\ker N^i$ is the null space of the matrix N^i and $\operatorname{Im} B_f$ is the image of B_f. Notice that $j \le q - 1$.

Definition 2 (*[16]*). A regular pencil $\lambda\bar{E} - \bar{A}$ is in *standard form* if there exist scalars α and β such that $\alpha\bar{E} + \beta\bar{A} = I$, where I is the identity matrix.

By definition, for any regular pencil $\lambda E - A$ there always exists a scalar μ such that $\det(\mu E - A) \neq 0$. Taking any such μ and multiplying (1a) on the left by $L = (\mu E - A)^{-1}$ gives

$$\bar{E}\dot{x} = \bar{A}x + \bar{B}u. \tag{3}$$

It is not difficult to verify that the pencil $\lambda\bar{E} - \bar{A}$ is in standard form for $\alpha = \mu$ and $\beta = -1$. The representation (3) is called a standard form of the regular implicit system (1a) [16]. Thus, for regular systems the assumption of a standard form is always without loss of generality. Also, since (1), (2) and (3) are restricted equivalent systems [12], we have $\Lambda(E, A) = \Lambda(I, A_s) = \Lambda(\bar{E}, \bar{A})$.

Recall that a single-input–single-output LTI regular implicit system of the form (1) has the transfer function [12]

$$g(s) = C(sE - A)^{-1}B = \frac{\eta(s)}{\delta(s)}, \tag{4}$$

where the polynomials $\delta(s) = \det(sE - A)$ and $\eta(s)$ and are the denominator and the numerator after zero–pole cancellation. We define the relative degree of (1) as $r = \deg\delta(s) - \deg\eta(s)$.

Now, consider the rational function $\pi \in \mathbb{C}(s)$ given by [17,12]

$$\pi(s) = \frac{1}{\mu - s}. \tag{5}$$

Strictly speaking, since π is not bijective, its inverse does not exist. However, we define $\pi^{-1} \in \mathbb{C}(s)$ as $\pi^{-1}(s) = \mu - 1/s$. Also, we agree that $\pi(\infty) = 0$.

Theorem 1 (*[12]*). *Consider a regular system (1) written in standard form. Let $g(s)$ be its transfer function. For $\tau = \pi(s)$ we have*

$$g(\pi^{-1}(\tau)) = C(\pi^{-1}(\tau)\bar{E} - \bar{A})^{-1}\bar{B} = \tau\bar{g}(\tau)$$

with $\bar{g}(\tau) = C(\tau I - \bar{E})^{-1}\bar{B}$.

Let us now turn to the questions of stability and controllability.

Theorem 2 (*[12,18]*). *The regular implicit system (1a) is stable if and only if $\Lambda(E, A) \subset \mathbb{C}^-$, where $\mathbb{C}^-$ represents the open left-half complex plane.*

We shall now introduce the concept of reachable state and characterize the set of all possible states reachable from a zero initial condition. This turns out to be important when distinguishing the different notions of controllability in regular implicit systems.

Definition 3. For a regular implicit system of the form (2), a vector $x_r \in \mathbb{R}^n$ is said to be *reachable* if there exists an initial condition $x_s(0)$, an input $u(\cdot) \in \mathcal{C}^j$, and some $t_1 > 0$ such that $\begin{bmatrix} x_s^\top(t_1) & x_f^\top(t_1) \end{bmatrix} = x_r^\top$.

Let $X_t(x_{s0})$ be the set of reachable states at time t from the initial condition $x_s(0) = x_{s0}$. Denote by $X_t = \bigcup_{x_{s0}\in\mathbb{R}^{n_1}} X_t(x_{s0})$ the set of reachable states at time t from all admissible initial conditions.