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Input-to-state stability for cascade systems with multiple invariant sets

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ABSTRACT

In a recent paper Angeli and Efimov (2015), the notion of Input-to-State Stability (ISS) has been generalized for systems with decomposable invariant sets and evolving on Riemannian manifolds. In this work, we analyze the cascade interconnection of such ISS systems and we characterize the finest possible decomposition of its invariant set for three different scenarios: 1. the driving system exhibits multistability (convergence to fixed points only); 2. the driving system exhibits multi-almost periodicity (convergence to fixed points as well as periodic and almost-periodic orbits) and the driven system is assumed to be incremental ISS; 3. the driving system exhibits multiperiodicity (convergence to fixed points and periodic orbits) whereas the driven system is ISS in the sense of Angeli and Efimov (2015). Furthermore, we provide marginal results on the backward/forward asymptotic behavior of incremental ISS systems and on the response of a contractive system under asymptotically almost-periodic forcing. Three examples illustrate the potentiality of the proposed framework.

1. Introduction

Input-to-State Stability (ISS) has been proven a very meaningful notion of stability and sensitivity to disturbances for nonlinear systems [1]. Apart from being a tool for the analysis, ISS has had a central role in the design of nonlinear feedback systems, with applications ranging from feedback redesign, small-gain theorems, tracking design, observers, and stabilization under saturated feedback. One of the major advances in this direction is the stabilization of nonlinear cascades, whose recursive application led to several constructive design methods such as backstepping and forwarding [2]. Indeed, in many cases of interest, the cascaded decomposition of the system under consideration is advantageous in providing the explicit stabilizing feedback law. Moreover, the ISS property behaves well under composition: a cascade of ISS system is again ISS, under suitable dissipation rates and gain functions of the driving/driven system, see [3].

Recently, a generalization of ISS theory for systems with decomposable invariant sets and evolving on Riemannian manifolds [4] has allowed the stability analysis in presence of inputs for a broader variety of systems exhibiting many dynamical behaviors of interests, such as multistability, periodic oscillations, almost global asymptotic stability, just to name a few. In this new setting, the decomposable invariant sets are no longer required to satisfy the Lyapunov stability requirement as long as they retain the global attractivity property and admit a decomposition without cycles, as specified in Definitions 2.1, 2.2 and 2.3 (basically no homoclinic nor heteroclinic orbits may exist).

Largely inspired by the applications in biological (see the mitogen-activated protein kinase (MAPK) as an example of cascade) as well as mechanical networks, in this work we study nonlinear cascades of systems belonging to the class described above, so that the novel generalized ISS theory can be applied. Not surprisingly, the ISS property is still conserved under cascade interconnection, under the implicit requirement to specify a compact invariant set for the cascade which is globally attractive and admits a decomposition without cycles. In particular, we characterize the finest possible decomposition of such invariant set in three different scenarios. In the first one, the driving system is assumed to exhibit multistable behavior, that is asymptotic convergence of all trajectories to fixed points only; the results provided by Thieme [5] for asymptotically autonomous semiflows turn out to be crucial in the analysis of this setting. In the second scenario, the driving system is assumed to have fixed points as well as periodic orbits and almost-periodic attractors (multi-almost periodicity), whereas the driven system is assumed to satisfy the incremental ISS property [6]. Indeed, incremental ISS is a very natural option for the analysis of this scenario. In the third scenario, the incremental ISS requirement for the driving system is relaxed to only ISS in the sense of [4]. It is within latter scenario that inferring ISS of the

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cascades comprises particularly novel results concerning the so-called converging-input-converging-state (CICS) [7,8] for systems under asymptotically periodic forcing.

The rest of the paper is organized as follows. Section 2 introduces the notion of decomposable invariant set and the class of cascade systems under consideration. Sections 3 and 4 respectively address the first and second scenarios (multistability and multi-almost periodicity). The third scenario is studied in Section 5. Section 6 collects examples for all three aforementioned scenarios. Final remarks are collected in Section 7.

Notation. Symbol $\mathfrak{d}(w_1, w_2)$ denotes the Riemannian distance between $w_1, w_2 \in M_w$. For a point $w \in M_w$, and for a subset $S \subset M_w$, the set-point distance is defined as:

$$|w|_S = \inf_{a \in S} \mathfrak{d}(w, a).$$

We define the equivalent of the infinity norm of the distance of signals taking values on M_w as follows:

$$\mathfrak{d}_{[a,b]}(X_1(\cdot), X_2(\cdot)) := \sup_{t \in [a,b]} \mathfrak{d}\left(X_1(t), X_2(t)\right).$$

Notation $|\cdot|$ indicates the standard Euclidean norm. For a measurable function $d : \mathbb{R}_+ \to \mathbb{R}^m$ we define its infinity norm over the time interval $[t_1, t_2]$ as $\left\|d_{[t_1,t_2]}\right\| = \text{esssup}_{t_1 \le t \le t_2} |d(t)|$, and denote $\|d\| := \left\|d_{[0,+\infty)}\right\|$.

2. Definitions and main assumptions

2.1. Decompositions and ISS for multistable systems

In this Section we will introduce the notion of decomposition of a compact invariant set of a nonlinear dynamical system. In fact, as pointed out in [4], the decomposition of a compact invariant set of a nonlinear system exhibiting neither homoclinic nor heteroclinic cycles plays a crucial role when claiming ISS. Moreover, it can be observed that such assumption automatically rules out a number of conservative systems (for instance, Hamiltonian systems).

Let M be an n-dimensional connected and geodesically complete Riemannian manifold without boundary. Let D be a closed subset of $\mathbb{R}^m$ containing the origin. Consider the system:

$$\dot{w}(t) = F(w(t), d(t)), \tag{1}$$

where $F(w, d) : M \times D \to T_x M$ is a locally Lipschitz continuous mapping with state w taking value in M and $d(\cdot)$ any locally essentially bounded and measurable input signal taking values in D. We denote by $W(t, w; d)$ the uniquely defined solution of (1) at time t fulfilling $W(0, w; d) = w$ under the input $d(\cdot)$.

The unperturbed system is defined by the following set of equations:

$$\dot{w}(t) = F(w(t), 0). \tag{2}$$

We assume that all solutions of (2) are complete[1] and that all (possibly empty) α- and ω-limit sets are compact.

Definition 2.1 ($\mathcal{W}$*-Limit Set*)**.** Let $\mathcal{W}_w \subset M$ be a compact invariant set containing all the α- and ω-limit sets of (2), i.e. $\alpha(w) \cup \omega(w) \subseteq \mathcal{W}_w$ for all $w \in M$. Then the set $\mathcal{W}_w$ is called an $\mathcal{W}$*-limit set for* (2).

Definition 2.2 (*Decomposition*)**.** Let $\mathcal{W}_w \in M$ be a compact and invariant set for (2). A *decomposition* of $\mathcal{W}_w$ is a finite, disjoint family of compact invariant sets $\mathcal{W}_{w,1}, \ldots, \mathcal{W}_{w,K}$ (the *atoms* of the decomposition) such that:

$$\mathcal{W}_w = \bigcup_{i=1}^{K} \mathcal{W}_{w,i}.$$

For an invariant set $\mathcal{W}_w$, its attracting and repulsing subsets are defined as follows:

$$\mathcal{A}(\mathcal{W}_w) = \left\{ w \in M_w : |W(t, w, 0)|_{\mathcal{W}_w} \to 0 \text{ as } t \to +\infty \right\},$$
$$\mathcal{R}(\mathcal{W}_w) = \left\{ w \in M_w : |W(t, w, 0)|_{\mathcal{W}_w} \to 0 \text{ as } t \to -\infty \right\}.$$

Define a relation on $\mathcal{W}_{w,i}$ and $\mathcal{W}_{w,j}$ by $\mathcal{W}_{w,i} \prec \mathcal{W}_{w,j}$ if $\mathcal{A}(\mathcal{W}_{w,i}) \cap \mathcal{R}(\mathcal{W}_{w,j}) \neq \emptyset$.

Definition 2.3 (*r-Cycle, 1-Cycle, Filtration*)**.** Let $\mathcal{W}_{w,1}, \ldots, \mathcal{W}_{w,K}$ be a decomposition of $\mathcal{W}_w$, then:

1. An *r-cycle* ($r \geq 2$) is an ordered r-tuple of distinct indexes $i_1, \ldots, i_r$ such that $\mathcal{W}_{w,i_1} \prec \cdots \prec \mathcal{W}_{w,i_r} \prec \mathcal{W}_{w,i_1}$.
2. A *1-cycle* is an index i such that $\left[\mathcal{R}(\mathcal{W}_{w,i}) \cap \mathcal{A}(\mathcal{W}_{w,i})\right] \setminus \mathcal{W}_{w,i} \neq \emptyset$.
3. A *filtration ordering* is a numbering of the $\mathcal{W}_{w,i}$ so that $\mathcal{W}_{w,i} \prec \mathcal{W}_{w,j} \Rightarrow i \leq j$.

Existence of an r-cycle for (2) with $r \geq 2$ is equivalent to existence of a heteroclinic cycle, and existence of a 1-cycle implies existence of a homoclinic orbit.

Assumption 2.4 (*No Cycle Condition*)**.** The autonomous system (2) is said to *satisfy the no-cycle condition* if it has an $\mathcal{W}$-limit set $\mathcal{W}_w$ as in Definition 2.1 that admits a finite decomposition without cycles, namely $\mathcal{W}_w = \bigcup_{i=1}^{K} \mathcal{W}_{w,i}$ for some non-empty disjoint compact sets $\mathcal{W}_{w,i}$, which form a filtration ordering of $\mathcal{W}_w$, as detailed in Definitions 2.2 and 2.3. Under the specified assumptions, the set $\mathcal{W}_w$ is said to *satisfy the no-cycle condition under the flow of* (2).

In the following, we recall a particular robustness notion for system (1) denoted as practical asymptotic gain (pAG) property [4].

Definition 2.5 (*pAG*)**.** System (1) is said to satisfy the *practical asymptotic gain (pAG)* property if there exist a class-$\mathcal{K}_\infty$ function η and $q \geq 0$ such that, for all $w \in M$ and all inputs $d(\cdot)$, solutions are defined for all $t \geq 0$ and the following holds:

$$\limsup_{t \to +\infty} |W(t, w; d)|_{\mathcal{W}_w} \leq \eta(\|d\|) + q. \tag{3}$$

If $q = 0$, then we say that the *asymptotic gain (AG)* property holds. If (3) holds with $q = 0$ and $\|d\| = 0$, we say that the systems (1) and (2) satisfy the *global zero-attractivity (0-GATT)* property.

The generalized notion of ISS for multistable systems in [4] replaces the Lyapunov stability requirement with Assumption 2.4 and is formalized as follows.

Definition 2.6. System (6) is said to be *ISS with respect to the input d and the invariant set* $\mathcal{W}$ if and only if $\mathcal{W}$ satisfies Assumption 2.4 and (6) has the AG property.

We will then consider a characterization of the ISS property in Definition 2.6 in terms of a Lyapunov dissipation inequality.

Definition 2.7 (*ISS-Lyapunov Function*)**.** A $\mathcal{C}^1$ function $V : M \to \mathbb{R}$ is a *practical ISS-Lyapunov function* for (6) if there exist $\mathcal{K}_\infty$ functions α_1, α, γ and $q \geq 0$ such that, for all $w \in M$ and all $d \in D$, the following holds:

$$\alpha_1(|w|_{\mathcal{W}_w}) \leq V(w) \tag{4}$$
$$DV(w)F(w, d) \leq -\alpha(|w|_{\mathcal{W}_w}) + \gamma(|d|) + q. \tag{5}$$

[1] Without loss of generality, system (1) can be made backward and forward complete by slowing down the dynamics with $\dot{w} = \frac{1}{1+|F(w,d)|_{\mathfrak{g}}} F(w, d)$, where $\mathfrak{g}$ denotes the Riemannian metric on M.