Systems & Control Letters 99 (2017) 85–89

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Simultaneously long short trading in discrete and continuous time

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ARTICLE INFO

Article history:
Received 1 June 2016
Received in revised form 18 November 2016
Accepted 28 November 2016
Available online 22 December 2016

Keywords:
Feedback-based stock trading
Technical trading rules
Simultaneously long short strategy
Sampled-data systems
Lévy processes

ABSTRACT

Simultaneously long short (SLS) feedback trading strategies are known to yield positive expected gain by zero initial investment for price processes governed by, e.g., geometric Brownian motion or Merton's jump diffusion model. In this paper, we generalize these results to positive prices with stochastically independent multiplicative growth and constant trend in discrete and continuous time as well as for sampled-data systems and show that in all cases the SLS strategies' expected gain does not depend on the price model but only on the trend.

1. Introduction

In this paper we extend recent results on control theory based strategies for stock trading. In general, traders who buy and sell stocks in order to make profit may use trading rules which tell them whether to invest or to disinvest in a specific stock. Such rules can be based, inter alia, on information on the underlying firm or solely on the stock's chart. For the latter type of strategies – usually called chartist strategies – control theoretic ideas have been systematically used in the last decades in order to derive so-called feedback trading rules and (performance) properties of this rules: based on [1], in [2,3] important properties of technical trading rules on geometric Brownian motion markets were derived, which were partially generalized to more complex models, e.g., Merton's jump diffusion model in [4,5]. Conceptually related work for stock prices modeled by geometric Brownian motions was done in [6–10]. Economical effects of technical analysis rules are analyzed in [11,12].

The basic idea of these rules, i.e., feedback trading strategies is rather simple: given trading times $t_0 < t_1 < \cdots < t_N$, instead of using the price path $p_t > 0$ for calculating the investment $I^{\ell}_{t_n}$ of trader ℓ at time t_n ($\mathbb{N}_0 \ni n \leq N$), feedback rules use the traders' own gain

$$g^{\ell}_{t_n} := \sum_{i=1}^{n} I^{\ell}_{t_{i-1}} \cdot \frac{p_{t_i} - p_{t_{i-1}}}{p_{t_{i-1}}} \tag{1}$$

based on the past investments $I_0, \dots, I_{n-1}$ and implement a feedback loop $I^{\ell}_{t_n} := f(g^{\ell}_{t_n})$ between investment and gain. Proceeding this way, the price process can be treated like a disturbance variable. Note that the investment can be positive (usually called *long*) as well as negative (*short*); likewise, the gain can be positive or negative. Investing short leads to a positive gain if prices fall.

The big question is: how to chose the function f? One possibility is to choose f as an affine linear function

$$I^L_t = I^*_0 + Kg^L_t \tag{2}$$

where $I^*_0 > 0$ is the initial investment and $K > 0$ is the feedback parameter. Since this is a long investing rule, that means it makes money if the prices rise, in a continuous time setting we call this rule linear long feedback trading strategy. Another choice is the analogous short rule

$$I^S_t = -I^*_0 - Kg^S_t$$

where g^S_t is the short rule's gain which is positive if prices are falling.[1] But since it is unrealistic that a trader knows whether prices are rising or falling it might be reasonable to choose the following simultaneously long short (SLS) strategy:

$$I^{SLS}_t = I^L_t + I^S_t.$$

For the reason of readability we write I_t and g_t instead of I^{SLS}_t and g^{SLS}_t, resp. Note – and this is very important – that g^L_t and g^S_t and I^L_t and I^S_t are still evaluated separately in order to determine the

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[1] We note that the names "long" and "short" here are true only for the continuous time version of these strategies. Indeed, in a discrete time setting it might happen that the long trader becomes a short trader and vice versa.

http://dx.doi.org/10.1016/j.sysconle.2016.11.011
0167-6911/

feedback strategy and that the initial investment of the SLS strategy is always zero ($I_0 = I_0^L + I_0^S = I_0^* - I_0^* = 0$).

The SLS trading strategy is in the focus of our research since there are some interesting results in the literature: in [1] it is shown that the gain of the SLS rule is positive for continuously differentiable prices which means the SLS strategy offers an arbitrage opportunity. In [2,3] it is shown that the SLS rule's expected gain is positive for prices following a geometric Brownian motion which has the property:

$$\mathbb{E}\left[\frac{dp(t)}{p(t)}\right] = \mu \tag{3}$$

with $\mu > -1$ being the trend. In particular it is shown that

$$\mathbb{E}[g_t] = \frac{I_0^*}{K}\left(e^{K\mu t} + e^{-K\mu t} - 2\right) \tag{4}$$

which is positive for all $t > 0$ and $\mu \neq 0$. In [4] this is generalized to prices that follow Merton's jump diffusion model, i.e., if the model parameters fulfill (3) the expected gain fulfills (4). In [5], this property is shown for a whole set of price models, called essentially linearly representable prices. These include geometric Brownian motions and Merton's jump diffusion model. That means, for many price models it could be shown that the expected gain is positive while the initial investment is zero. Similar results were obtained for the strategies introduced in [7–9] for prices governed by geometric Brownian motions.

In the work at hand, we further generalize the results for the SLS rule by showing that this property – positive expected gain with zero initial investment – holds for all discrete and continuous price processes with independent multiplicative growth and constant trend. For example, a exponentiated Lévy process fulfills this properties. Furthermore, we show our results in the practically more realistic discrete time setting and give a closed formula for the expected gain of the SLS strategy. In this context, we clarify the relation between the discrete time or sampled-data setting considered in this paper and the continuous time setting used in most of the literature on feedback trading. In particular, and in contrast to sampled-data implementations of other controllers known in the literature [13–15], we show that when the sampled controller is applied to a continuous time process then there is no qualitative change in the performance of the closed loop properties, i.e., the property of positive expected gain is maintained for arbitrary sampling times $h > 0$, only the amount of the expected gain changes with the sampling time.

The paper is organized as follows: After an introduction to trading, SLS trading, and related work, the price processes of interest are defined and market requirements are presumed. In Section 3 a formula for the expected gain of the SLS trading strategy in discrete time is derived. In Section 4 the application of this trading strategy to a continuous time process as a sampled-data controller is analyzed and in Section 5 the limit for vanishing sampling times is computed and found to be consistent with the existing continuous time results in the literature. At the end, the paper is concluded and references are given.

2. Price processes and market requirements

Before analyzing the SLS strategy, we have to specify the price processes of interest and the time grid on which we define the price processes.

- Discrete Time Trading: at every point of time $t \in \mathcal{T} = \{0, h, 2h, \ldots, T\}$ with $T = Nh$ and $h > 0$, the trader has all information available up to t and adjusts his/her investment I_t.

Definition 1. Given $h > 0$ and $\mathcal{T}$ from above, the price processes of interest have the following properties:

- Stochastic Prices: the price process $(p_t)_{t\in\mathcal{T}}$ is a stochastic process.
- Positive Prices: the price p_t is positive for all $t \in \mathcal{T}$.
- Fixed Start Price: The start price $p_0 \in \mathbb{R}^+$ is deterministic.
- Independent Multiplicative Growth: for all $k \in \mathbb{N}$ and all $t_0 < t_1 < \cdots < t_k \in \mathcal{T}$ it holds:
$$p_{t_0}, \frac{p_{t_1}}{p_{t_0}}, \frac{p_{t_2}}{p_{t_1}}, \ldots, \frac{p_{t_k}}{p_{t_{k-1}}} \text{ are stochastically independent.} \tag{5}$$
- Constant Trend: the expected relative return is constant, i.e., there is $\mu_h > -1$ such that for all $t \in \mathcal{T} \setminus \{0\}$ it holds:
$$\mathbb{E}\left[\frac{1}{p_{t-h}} \cdot \frac{p_t - p_{t-h}}{h}\right] = \mu_h. \tag{6}$$
Note that this assumption is inspired by (3) and that it is equivalent to:
$$\mathbb{E}\left[\frac{p_t}{p_{t-h}}\right] = \mu_h h + 1. \tag{7}$$

Additionally, we need some basic market requirements which are similar to those in [2,4].

Definition 2. The following market requirements are presumed:

- Costless Trading: there are no additional costs associated with buying or selling an asset.
- Adequate Resources: the trader has enough financial resources so that all desired transactions can be executed.
- Trader as Price-Taker: the trader is not able to influence the asset's price, neither directly nor through buying or selling decisions. Note that in case $h > 0$ this value is not fixed but considered a parameter of the trader (determined by the trading frequency), this appears to be a contradiction to the definition of μ_h since the relative return in (6) may then depend on the trading frequency. We will see in Section 4, below, why this is not a contradiction.
- Perfect Liquidity: there is neither a gap between bid and ask price nor any waiting time for transaction execution.

Before analyzing the trading performance, we will have a closer look on the prices fulling above defined assumptions. At first, we will prove a lemma concerning the expected stock price. Note that the idea of the proof will be very helpful when analyzing the trading strategy, too.

Lemma 1. *For $t = nh$, a price process fulfilling Definition 1 has the expected value:*

$$\mathbb{E}[p_t] = p_0 \cdot z\left(\mu_h, \frac{1}{h}\right)^t$$

with $z(x, m) : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ given by $z(x, m) \mapsto \left(1 + \frac{x}{m}\right)^m$.

Proof. This can be proven by calculation using Definition 1:

$$\begin{aligned}\mathbb{E}[p_t] &= \mathbb{E}\left[p_0 \cdot \frac{p_h}{p_0} \cdot \frac{p_{2h}}{p_h} \cdots \frac{p_t}{p_{(n-1)h}}\right] \\ &= p_0 \cdot \prod_{i=1}^{n} \mathbb{E}\left[\frac{p_{ih}}{p_{(i-1)h}}\right] \\ &= p_0 \cdot (\mu_h h + 1)^n \;=\; p_0 \cdot \left((\mu_h h + 1)^{\frac{1}{h}}\right)^t.\end{aligned}$$

Now the definition of the function z proves the lemma. $\square$