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Research paper

Perturbative dynamics of stationary states in nonlinear parity-time symmetric coupler

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A B S T R A C T

We investigate the nonlinear parity-time (PT) symmetric coupler from a dynamical perspective. As opposed to linear PT-coupler where the PT threshold dictates the evolutionary characteristics of optical power in the two waveguides, in a nonlinear coupler, the PT threshold governs the existence of stationary points. We have found that the stability of the ground state undergoes a phase transition when the gain/loss coefficient is increased from zero to beyond the PT threshold. Moreover, we found that instabilities in initial conditions can lead to aperiodic oscillations as well as exponential growth and decay of optical power. At the PT threshold, we observed the existence of a stable attractor under the influence of fluctuating gain/loss coefficient. Phase plane analysis has shown us the presence of a toroidal chaotic attractor. The chaotic dynamics can be controlled by a judicious choice of the waveguide parameters.

1. Introduction

Bender and Boettcher's pioneering work [1] on a class of non-Hermitian Hamiltonian paved the way for new developments in the foundational studies of quantum mechanics [2–6]. They showed that such Hamiltonians possess a real eigenspectra as long it respects the criteria of *PT* (Parity and Time Reversal) symmetry. In general, the Hamiltonian $H = -\frac{1}{2}\frac{d^2}{dx^2} + V\ (x)$ is said to be *PT* symmetric if the potential function satisfy $V(x) = V^*(-x)$. Such Hamiltonians possess a real eigen-spectrum. But if the imaginary component of $V(x)$ exceeds a certain threshold, the eigenspectrum ceases to be real resulting in spontaneous symmetry breaking [7].

In recent times, optics has proved to be a fertile ground for the investigation of *PT* symmetry both in linear as well as nonlinear systems. It was Ruschhaupt, Delgado and Muga [8], who first proposed the idea in 2005 in the context of planar slab waveguides. Moreover, the isomorphism of the paraxial equation of diffraction [8] with Schrodinger's wave equation presented a feasible way to explore *PT* symmetry in the field of optics provided one can appropriately synthesize the refractive index profile of the system to satisfy, $n(x) = n^*(-x)$. This analogy enabled researchers to observe the first experimental evidence of *PT* symmetry in optical waveguide structures [7]. Since then there has been numerous works on *PT* symmetry in optics, both in experimental as well as theoretical settings. PT Symmetry is studied in various contexts such as: Bragg solitons in nonlinear PT-symmetric periodic potential [9], continuous and discrete Schrodinger systems with PT-symmetric nonlinearities [10–12], bright and dark solitons and existence of optical rogue waves [13–19], modulation instability in nonlinear PT-symmetric structures [20,21], optical oligomers [22–29], optical mesh lattices [30–33], unidirectional invisibility

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[34], non-reciprocity and power oscillations [35,36], field propagation in linear and nonlinear stochastic *PT* coupler [37], optical mode conversion and transmission on photonic circuits [38] and so on.

In coupled waveguide systems, the *PT* phase transition is characterized by exponential growth and decay of optical power. Such systems have been studied in great detail [22]. The equations governing such systems can be analytically solved if the system is devoid of any form of nonlinearity. But in the presence of nonlinearity, analytical solution is not possible and prior assumptions are required. For instance, in Ref. [22], the system was studied taking stationary waves into consideration, whereas in Ref. [23], Stokes' parameters were used to study the conserved quantities. In the same line of research, this work aims to study the nonlinear *PT* symmetric dimer from a dynamical point of view. We have considered a waveguide coupler as our 'dimer' system. A thorough stability analysis of the fixed or stationary points in the system is carried out. This gives us a clearer and detailed interpretation of the dynamics subjected to different initial conditions. In our discussion, we will use the terms fixed points and stationary states interchangeably.

The article is organized as follows. In Section II, the theoretical model is described briefly. Section III presents and discusses the stability analysis of the ground state of the coupler below and above the *PT* threshold. It also discusses the non-zero stationary states of the configuration in the unbroken regime and at the phase transition point followed by conclusion in Section IV.

2. The model

The *PT* symmetric nonlinear coupler is a configuration consisting of two waveguides in close proximity so as to facilitate the transfer of optical power from one waveguide to the other via evanescent coupling. One waveguide can amplify the input optical signal and the other can attenuate the signal by the same proportion. The equations governing the dynamics of such a configuration are given by [23]:

$$\begin{aligned} i\frac{da_1}{dz} &= i\gamma a_1 + Ca_2 + |a_1|^2 a_1 \\ i\frac{da_2}{dz} &= -i\gamma a_2 + Ca_1 + |a_2|^2 a_2 \end{aligned} \tag{1}$$

Here, a_1 and a_2 are the field amplitudes and γ characterizes the gain/loss in the two channels and C is the coupling constant. Both waveguides portray Kerr nonlinearity of equal strength.

In the absence of Kerr nonlinearity, the *PT* threshold is given by $\gamma_{th} = C$. But adding the nonlinearity changes the entire dynamics of the system. The reason is that once the system is modified with the inclusion of nonlinear terms, the initial conditions will play a major role in the dynamics of optical power evolution [28]. It must be noted here that the *PT* threshold of the linear coupler will be used as a reference point to study the stability analysis.

3. Stability analysis and discussion

We first consider the ground state of the coupler defined by: $a_1 = a_2 = 0$. This set of initial conditions corresponds to unexcited waveguides. To ascertain the stability of the ground state, we expand the differential equations using the prescription $a_1 = x_1 + iy_1$ and $a_2 = x_2 + iy_2$. Eq. (1) can then be re-written as follows:

$$\dot{x}_1 = \gamma x_1 + Cy_2 + \left(x_1^2 + y_1^2\right)y_1 \tag{2a}$$

$$\dot{y}_1 = \gamma y_1 - Cx_2 - \left(x_1^2 + y_1^2\right)x_1 \tag{2b}$$

$$\dot{x}_2 = -\gamma x_2 + Cy_1 + \left(x_2^2 + y_2^2\right)y_2 \tag{2c}$$

$$\dot{y}_2 = -\gamma y_2 - Cx_1 - \left(x_2^2 + y_2^2\right)x_2 \tag{2d}$$

The linearization Jacobian is given by

$$J = \begin{bmatrix} \gamma + 2x_1y_1 & x_1^2 + 3y_1^2 & 0 & C \\ -\left(3x_1^2 + y_1^2\right) & \gamma - 2x_1y_1 & -C & 0 \\ 0 & C & -\gamma + 2x_2y_2 & x_2^2 + 3y_2^2 \\ -C & 0 & -(3x_2^2 + y_2^2) & -\gamma - 2x_2y_2 \end{bmatrix} \tag{3}$$

The Jacobian eigenvalues are calculated to be $\lambda = \pm\sqrt{\gamma^2 - C^2}$. For $\gamma < C$, all eigenvalues of the Jacobian are purely imaginary indicating that the ground state is a non-hyperbolic fixed point [39]. Linear stability analysis fails if the fixed point under consideration is non-hyperbolic [40]. In mathematical terms, if all the eigenvalues are purely imaginary, the fixed point is classified as non-hyperbolic. In such a case, numerical solution of the system, under a suitably chosen perturbation, reveals the exact nature of the fixed point. On the other hand, if one or some of the eigenvalues contain non-zero real part the fixed point is categorized as hyperbolic. In such cases, linear stability analysis is sufficient. Above the *PT* threshold, the