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Research paper

Conservation laws for certain time fractional nonlinear systems of partial differential equations

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ABSTRACT

In this study, an extension of the concept of nonlinear self-adjointness and Noether operators is proposed for calculating conserved vectors of the time fractional nonlinear systems of partial differential equations. In our recent work (J Math Phys 2016; 57: 101504), by proposing the symmetry approach for time fractional systems, the Lie symmetries for some fractional nonlinear systems have been derived. In this paper, the obtained infinitesimal generators are used to find conservation laws for the corresponding fractional systems.

1. Introduction

The investigation of Lie symmetries and conservation laws for nonlinear partial differential equations (PDEs) plays an important role in the study of nonlinear physical phenomena. Over the past few years, great progress can be noticed in the research on fractional order differential equations (FDEs). The methods for symmetry analysis and conserved vectors have already been discussed for single time fractional PDEs [1,2]. Recently, the symmetry approach has been developed for systems of time fractional PDEs [3]. However, to the best of our knowledge, the idea of investigating conservation laws using symmetry generators has not been extended to the systems of time fractional PDEs. In this study, the method to investigate conservation laws is proposed as well as applied to some time fractional nonlinear systems. In our recent paper [3], the Lie symmetries for five time fractional nonlinear systems namely coupled Ito system, coupled Burgers equations, coupled KdV system, Hirota–Satsuma coupled KdV system, coupled Hirota equations have been obtained. Here, the derived symmetries are further used in calculating conserved vectors for the time fractional systems.

The conservation laws originate from physical principles such as conservation of mass, momentum and energy [4]. The conservation laws are also used for development of numerical methods [5], establishing existence and uniqueness of solu-

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tions [4], study of properties such as bi-Hamiltonian structures and recursion operators [6]. There are several generalizations of Euler–Lagrange's equations and Noether's theorem [7] corresponding to various definitions of fractional derivative to find conservation laws for fractional PDEs having fractional Lagrangians [8–11]. On the contrary, for FDEs not having Lagrangians, the derivation of conserved vectors is limited to only single time fractional PDEs [2,12,13]. The main aim of this study is to present a technique using Lie symmetry analysis for determining the conservation laws for time fractional systems of PDEs. For this purpose, the concept of nonlinear self-adjointness is extended for time fractional systems. Also, a generalization of fractional Noether operators is provided for the formulation of conservation laws for time fractional systems with the aid of new conservation theorem [14]. This new approach is applied to investigate conserved vectors of five time fractional nonlinear systems with fractional order $\alpha \in (0, 2)$.

The rest of the paper is organized as follows. In Section (2), the approach for deriving conservation laws for time fractional systems of PDEs is introduced. Section (3) deals with the application of proposed approach for investigating conservation laws for some time fractional nonlinear systems. The concluding remarks are given in the last Section.

2. Description of the approach

In this section, the technique to find conservation laws is proposed by investigating the nonlinear self-adjointness of the time fractional systems. Consider a time fractional system of the following form:

$$F_j(x, t, u_1, .., u_m, \partial_t^\alpha u_1, .., \partial_t^\alpha u_m, u_{1,x}, .., u_{m,x}, ..) = 0, \quad j = 1, .., m, \tag{1}$$

with two independent variables (x, t) and m $(m > 1)$ dependent variables $(u_1, \ldots, u_m)$. Here, $\partial_t^\alpha u_j = \frac{\partial^\alpha u_j}{\partial t^\alpha}$ are the Riemann–Liouville fractional derivatives [3,15] of u_j of order α , $u_{j,x} = \frac{\partial u_j}{\partial x}$ and so on. Assume that the system (1) admits the Lie group of transformations with symmetry generators given by

$$V_i = \xi_i \partial_x + \tau_i \partial_t + \sum_{j=1}^{m} \eta_i^j \partial_{u_j}, \quad i = 1, \ldots, n, \tag{2}$$

where $\partial_{u_j} = \frac{\partial}{\partial u_j}$ is the first order partial derivative.

2.1. Nonlinear self-adjointness

The idea of nonlinear self-adjointness is very well known and well established [16–18] for the integer order PDEs. Recently, it is presented for single time fractional PDEs [2,13]. In this section, the concept of nonlinear self-adjointness is extended from single time fractional PDEs to the systems of time fractional PDEs. The formal Lagrangian can be written in the following form:

$$\mathcal{L} = \sum_{j=1}^{m} p_j(x, t)(F_j), \tag{3}$$

where $p_j(x, t)$ are the new dependent variables. The adjoint equations are defined as follows:

$$F_j^* \equiv \frac{\delta \mathcal{L}}{\delta u_j} = 0, \quad j = 1, \ldots, m. \tag{4}$$

Here, $\frac{\delta}{\delta u_j}$ represent the Euler–Lagrange operators defined by

$$\frac{\delta}{\delta u_j} = \frac{\partial}{\partial u_j} + (D_t^\alpha)^* \frac{\partial}{\partial (D_t^\alpha u_j)} + \sum_{k=1}^{\infty} (-1)^k \mathrm{D}_{i_1} \mathrm{D}_{i_2}, \ldots, \mathrm{D}_{i_k} \frac{\partial}{\partial (u_j)_{i_1, i_2, \ldots, i_k}}, \tag{5}$$

where D_i is the total derivative operator, D_t^α is the Riemann–Liouville fractional derivative operator [3,15]. Also, $(D_t^\alpha)^*$ is the adjoint operator defined by

$$(D_t^\alpha)^* = (-1)^n I_c^{n-\alpha}(D_t^n) = {}_t^C D_c^\alpha, \tag{6}$$

with $I_c^{n-\alpha}$ is the right-hand sided fractional integral operator of order $n - \alpha$ given by

$$I_c^{n-\alpha} f(x, t) = \frac{1}{\Gamma(n-\alpha)} \int_t^c \frac{f(x, s)}{(s-t)^{1+\alpha-n}} \mathrm{d}s, \quad \text{for} \quad n = [\alpha] + 1. \tag{7}$$

Also, ${}_t^C D_c^\alpha$ is the right-hand sided Caputo fractional differential operator [13,15] of order α.

The time fractional system (1) is called nonlinearly self-adjoint if the adjoint Eq. (4) are satisfied for all solutions of (1) upon the following substitutions:

$$p_j = \psi_j(x, t, u_1, .., u_m), \quad j = 1, .., m, \tag{8}$$

satisfying $\psi_j \neq 0$ for at least one j.