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Research paper

Ginzburg-Landau approximation for self-sustained oscillators weakly coupled on complex directed graphs

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ABSTRACT

A normal form approximation for the evolution of a reaction-diffusion system hosted on a directed graph is derived, in the vicinity of a supercritical Hopf bifurcation. Weak diffusive couplings are assumed to hold between adjacent nodes. Under this working assumption, a Complex Ginzburg–Landau equation (CGLE) is obtained, whose coefficients depend on the parameters of the model and the topological characteristics of the underlying network. The CGLE enables one to probe the stability of the synchronous oscillating solution, as displayed by the reaction-diffusion system above Hopf bifurcation. More specifically, conditions can be worked out for the onset of the symmetry breaking instability that eventually destroys the uniform oscillatory state. Numerical tests performed for the Brusselator model confirm the validity of the proposed theoretical scheme. Patterns recorded for the CGLE resemble closely those recovered upon integration of the original Brussellator dynamics.

1. Introduction

Many real-life phenomena can be ultimately described in terms of mutually interacting entities which can occasionally give rise to collective behaviors [1,2]. The emerging patterns may play important functional roles, as it is for instance the case for biochemical processes [3] and ecological applications [4–6]. Synchrony is among the most striking example of self-organized dynamics [7]. It is encountered in a wide gallery of natural systems, think to the beating of the heart [8] and the firing of the firefly [9]. It also plays a role of paramount importance for the correct functioning of man-designed technology, as e.g. in power grids [10,11].

Synchronization of self-sustained oscillations is a particularly rich field of investigation. Mathematically, self-sustained oscillations correspond to stable limit cycles in the state space of an autonomous continuous-time dynamical system. Oscillators can be embedded in continuum space, being subject to diffusive couplings. The ensuing reaction-diffusion system displays synchronous oscillations, which, under specific conditions, prove robust to external perturbations. Each oscillator can alternatively occupy a node of a complex network [12–15], a generalization that opens up the perspective to tackle a large plethora of problems that deal with a discrete and heterogeneous hosting support [15–17].

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For continuous reaction-diffusion systems near the supercritical Hopf bifurcation, one can obtain a simplified, normal form description of the dynamics in terms of an associated Ginzburg–Landau equation (CGLE) [7,18,19]. This is a reduced picture which provides an accurate representation of the original model, while allowing for analytical progress to be made. From a more general perspective, it is also important to classify minimal, though effective descriptive frameworks that could guide in the search for universal traits that happen to be shared by multispecies models, besides their intrinsic degree of inherent specificity. Simplified schemes that exemplify an exact underlying dynamics can be effectively employed to shed light on the onset of spatio-temporal chaos and propagation of nonlinear waves. Externally imposed non homogeneous perturbation can, for instance, break the synchrony of the oscillations as displayed by the original reaction-diffusion system, or equivalently its CGLE analogue, so materializing in a colorful density patterns that sustain spatio-temporal propagation. The formal link between continuous reaction diffusion-systems and the CGLE was established in the pioneering work by Kuramoto [18], exploiting a multiple-scale perturbative analysis. More recently the analysis has been extended by Nakao [20] to the relevant setting where the reaction-diffusion system is made to evolve on a symmetric, hence undirected, network. As remarked in [21], directionality matters and can seed the emergence of non trivial collective dynamics which cannot manifest when the scrutinized system is made to evolve on a symmetric discrete support. Motivated by this finding, we considered in [22] the dynamics of a reaction-diffusion system defined on a directed graph which displays a stable fixed point and obtained an effective description for the evolution mode triggered unstable, just above the threshold of criticality. The analysis exploits a multiple time-scale analysis and eventually yields a Stuart–Landau for the amplitude of the unstable mode, whose complex coefficients reflect the topology of the network, the factual drive to the instability.

In this paper we aim at following the similar strategy of [22] to derive an approximate equation for the evolution of a reaction diffusion system on a directed (and balanced) graph, in the vicinity of a supercritical Hopf bifurcation. As a matter of fact we will assume weak the strength of the coupling that links adjacent nodes. In doing so, we will generalize the work of Nakao [20] to the interesting setting where asymmetry in the couplings needs to be accommodated for and, at the same time, reformulate the classical work of Kuramoto [18], on a discrete spatial backing. To anticipate our findings, and at variance with the analysis reported in [22], we will finally obtain a CGLE as a minimal description for the dynamics of the self-sustained oscillators coupled on a complex and asymmetric graph. The obtained CGLE enables one to analytically probe the stability of the synchronous uniform state, as displayed by the reaction-diffusion system. Specifically, it allows to determine the parameters setting that instigates a symmetry breaking transition to non-uniform patterns. Numerical tests made for the Brusselator model, here assumed as a reference model for its pedagogical interests, will confirm the adequacy of the proposed approximate scheme.

2. Diffusive oscillators on networks

We will here consider a generic two dimensional reaction-diffusion system and label with $\mathbf{x}_j(t) = (\phi_j, \psi_j)^T$ for $j = 1, \ldots, N$ the two-dimensional real vector of the concentrations. The index j refers to the node of the network to which the selected components refer to. N stands for the size of the network, i.e. the total number of nodes. The only further assumption that we shall make is the existence of self-sustained oscillations for the system under scrutiny and for this reason we will point to $\mathbf{x}_j(t)$ as to the oscillators' variables. The dynamics of the system is hence described by the following differential equation

$$\dot{\mathbf{x}}_j = \mathbf{F}(\mathbf{x}_j, \boldsymbol{\mu}) + \mathbf{D}\sum_{k=1}^{N} \Delta_{jk}\mathbf{x}_k \tag{1}$$

where the two-dimensional nonlinear function $\mathbf{F}$ specifies the reaction terms: it depends on the local concentrations of the species $\mathbf{x}_j$ and on $\boldsymbol{\mu}$ which is a vector of arbitrary dimension that gather together the parameters of the model. The second term represents the diffusive coupling: $\mathbf{D} = K\mathrm{diag}(D_\phi, D_\psi)$ denotes the diagonal matrix of the diffusion coefficients. K is a constant parameter that set the strength of the coupling. As we will make clear in the following the perturbative analysis that we shall develop, hold for $K << 1$. In Eq. (1), $\boldsymbol{\Delta}$ is the Laplacian matrix whose elements read $\Delta_{ij} = A_{ij} - \delta_{ij}k_i$. Here we focus on directed networks, thus the adjacency matrix $\mathbf{A}$ is not symmetric. Using standard notations, $A_{ij} = 1$ if a link exists that goes from node i to node j. Otherwise, $A_{ij} = 0$. k_i is the number of outgoing edges from node i, δ_{ij} is the Kronecker's delta. We assume that system (1) admits a homogeneous equilibrium point that we here label $\mathbf{x}^* = (\phi^*, \psi^*)^T$. This request implies dealing with a balanced network, namely a network where the outgoing and incoming connectivities are equal.

We additionally require that $\mathbf{x}^*$ undergoes a Hopf bifurcation for $\boldsymbol{\mu} = \boldsymbol{\mu}_0$. Accordingly, the Jacobian matrix associated to system (1) has a pair of imaginary eigenvalues $\pm i\omega_0$. Slightly above the supercritical Hopf bifurcation, $\mathbf{x}^*$ becomes unstable, the reaction-diffusion system admits an time dependent homogeneous solution. This is the uniform state obtained by replicating on each node of the network and in complete synchrony, the limit cycle displayed by the system in its a-spatial limit ($K = 0$). The spatial coupling, sensitive to tiny non homogeneities, which configure as injected perturbation, can eventually destabilized the uniform synchronous equilibrium. When diffusion is small ($K = \epsilon^2 << 1$), the method of multiple timescales [18] constitutes a viable strategy to characterize the nonlinear evolution of the perturbation and hence elaborate on the stability of the time-dependent uniform periodic solution.

To this end, inspired by the analysis carried out in [20], we introduce small inhomogeneous perturbations, $\delta\phi_j$ and $\delta\psi_j$, to the uniform equilibrium point, namely $(\phi_j, \psi_j) = (\phi^*, \psi^*) + (\delta\phi_j, \delta\psi_j)$ for $j = 1, \ldots, N$. We then substitute this