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Research paper

$K(m, n)$ equations with fifth order symmetries and their integrability

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ABSTRACT

For $K(m, n)$ equation $u_t = D_x^3(u^n) + \alpha D_x(u^m)$, all non-degenerate $(n \neq 0)$ cases admitting fifth order symmetries are identified, including $K(m_1, 1)$, $K(m_2, -1/2)$ and $K(m_3, -2)$, where $m_1 = 0, 1, 2, 3$, $m_2 = -1/2, 0, 1, 3/2$ and $m_3 = -2, -1, 0, 1$. For five less studied cases, namely $K(0, -2)$, $K(-1, -2)$, $K(-2, -2)$, $K(-1/2, -1/2)$ and $K(3/2, -1/2)$, bi-Hamiltonian structures are established through their invertible links with some famous integrable equations. Hence, all cases, having fifth order symmetries, of $K(m, n)$ equation are integrable in the bi-Hamiltonian sense. As an interesting observation, their Hamiltonian operators are linearly combinations of D_x, D_x^3, $uD_x + D_xu$ and $D_xuD_x^{-1}uD_x$, basic ingredients in the bi-Hamiltonian theory of Korteweg-de Vries and modified Korteweg-de Vries equations.

1. Introduction

To understand the role of nonlinear dispersion in wave motion, Rosenau and Hyman introduced the so-called $K(m, n)$ equation [1]

$$u_t + D_x^3(u^n) + D_x(u^m) = 0, \quad (n \text{ and } m \text{ are free parameters.}) \tag{1}$$

where $u = u(x, t)$,

$$D_x = \frac{\partial}{\partial x} + \sum_{j=0} u_{(j+1)x} \frac{\partial}{\partial u_{jx}}$$

denotes the total derivative with respect to x and u_{jx} stands for $\partial^j u/\partial x^j$. Some celebrated integrable models are special cases of $K(m, n)$ equation, for instance, $K(2, 1)$, $K(3, 1)$ and $K(0, -1/2)$ are nothing but the Korteweg-de Vries (KdV) equation, the modified KdV equation and the Harry Dym equation, respectively. As illustrated by traveling wave solutions to $K(2, 2)$ equation [1], the notable feature of Eq. (1) is that it may possess solitary wave solutions with compact support, or compactons. This novelty attracts much interest from both mathematicians and physicists, and arouses them to study Eq. (1) from different points of view.

In his follow-up work [2], Rosenau, through Lagrange-type maps, identified some integrable cases of Eq. (1), including $K(-1, -2)$, $K(-2, -2)$ and $K(3/2, -1/2)$, which are linked to the (modified) KdV equation. More sophisticated, Lagrange maps connect N-compactons with N-solitons, and make interaction of compactons clear. It was pointed out that compactons

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are not expected to interact elastically unless the compacton-carrying equation is integrable [2]. Therefore, integrable cases bear special interests among the whole $K(m, n)$ family. Many efforts were made to locate more integrable cases of $K(m, n)$ equation with different methods, such as Painlevé analysis [3], Whalquist-Estabrook prolongation technique [4] and others, but those results were not always consistent. For instance, $K(1/2, -1/2)$ equation was claimed to be equivalent to the potential modified KdV equation [3], and was also shown not to admit any non-trivial prolongation structures [4]. These two judgments conflict with each other since the former is an indication of integrability, while the latter a prediction of non-integrability.

To clarify those confusions, we will make our own effort to identify integrable cases of $K(m, n)$ equation. With more generality, we add a new parameter α to the original Eq. (1), and consider the model

$$u_t = D_x^3(u^n) + \alpha D_x(u^m), \tag{2}$$

which is still referred to as $K(m, n)$ equation.

In the theory of infinite dimensional integrable systems [5], there is no universal definition of integrability for nonlinear differential equations. Among various (maybe equivalent) conceptions of integrability, existence of infinitely many symmetries (or conservation laws) is essential and strongly indicates other integrable properties. One necessary condition leading to infinitely many symmetries is stated as admitting higher order *infinitesimal symmetries* (abbreviated to symmetry provided no confusions will arise) [6,7], and it is even sufficient in some situations. Therefore, *admitting higher order symmetries could be adopted as a precondition to test integrability.* One aim of this paper is to find out all casas of $K(m, n)$ equation (2), which have fifth order symmetries, and then give a complete list of integrable cases of $K(m, n)$ equation in the symmetry sense. Associated with symmetries and conserved quantities closely, a spectacular feature of a certain integrable Hamiltonian system is due to its bi-Hamiltonian structure [8], which provides us with an effective tool to describe the whole hierarchy of symmetries, or the family of conserved quantities. Another purpose of this paper is to establish bi-Hamiltonian theory for all integrable cases of $K(m, n)$ equation (2).

This paper is organized as follows. In Section 2, all cases, admitting fifth order symmetries, of $K(m, n)$ equation (2) are identified by means of solving the corresponding linearized symmetry equation. Their integrability is shortly commented at the end of this section. In Section 3, quasi-linear integrable cases of $K(m, n)$ equation (2) are changed into semi-linear equations via reciprocal transformations defined on the basis of two simple conservation laws. Furthermore, their bi-Hamiltonian structures are established by applying those transformations to the well-known bi-Hamiltonian structure of (modified) KdV equation. Some concluding remarks are given in the last section.

2. Cases admitting fifth order symmetries

A smooth function $Q(x, t, u, u_x, \cdots, u_{lx})$ is a lth order infinitesimal symmetry of $K(m, n)$ equation (2) if $\partial Q/\partial u_{lx} \neq 0$ and

$$D_tQ - nD_x^3\left(u^{n-1}Q\right) - \alpha m D_x\left(u^{m-1}Q\right) = 0, \tag{3}$$

where D_t stands for the total derivative with respect to t on solutions to $K(m, n)$ equation, and is defined as

$$D_t = \frac{\partial}{\partial t} + \sum_{j=0} D_x^j\Big(D_x^3(u^n) + \alpha D_x(u^m)\Big)\frac{\partial}{\partial u_{jx}}.$$

We want to determine values of n, m and α so that the corresponding equation admits fifth order symmetries. To this end, substituting an arbitrary smooth function $Q(u, u_x, \cdots, u_{5x})$ into the symmetry condition (3), we will obtain a series of constraints on the function Q and three parameters m, n and α. Solving these constraints yields some special cases of $K(m, n)$ equation and their fifth order symmetries.

It is straightforward to have

$$D_tQ - nD_x^3\left(u^{n-1}Q\right) - \alpha m D_x\left(u^{m-1}Q\right) = n\left(5\left(D_xu^{n-1}\right)\frac{\partial Q}{\partial u_{5x}} - 3u^{n-1}\left(D_x\frac{\partial Q}{\partial u_{5x}}\right)\right)u_{7x} + G(u, u_x, u_{xx}, \cdots, u_{6x}),$$

where G denotes a smooth function of its arguments, which implies the first constraint on Q to be a symmetry of $K(m, n)$ equation, i.e.

$$5(D_xu^{n-1})\frac{\partial Q}{\partial u_{5x}} - 3u^{n-1}\left(D_x\frac{\partial Q}{\partial u_{5x}}\right) = 0.$$

Solving it, we have $\partial Q/\partial u_{5x} = u^{5(n-1)/3}$, which means all possible fifth order symmetries independent of x and t should take the form

$$Q \equiv u^{5(n-1)/3}u_{5x} + \bar{Q}(u, u_x, u_{2x}, u_{3x}, u_{4x}).$$

Substituting this function into the symmetry condition (3), the coefficient of u_{6x} gives a constraint on $\bar{Q}$ which turns out to be

$$4(D_xu^{n-1})\frac{\partial \bar{Q}}{\partial u_{4x}} - 3u^{n-1}\left(D_x\frac{\partial \bar{Q}}{\partial u_{4x}}\right) + 20(n-1)u^{7(n-1)/3}(D_xu^{(n-4)/3}u_x) = 0.$$