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Research paper

Stochastic sensitivity analysis of the variability of dynamics and transition to chaos in the business cycles model

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ABSTRACT

A problem of mathematical modeling of complex stochastic processes in macroeconomics is discussed. For the description of dynamics of income and capital stock, the well-known Kaldor model of business cycles is used as a basic example. The aim of the paper is to give an overview of the variety of stochastic phenomena which occur in Kaldor model forced by additive and parametric random noise. We study a generation of small- and large-amplitude stochastic oscillations, and their mixed-mode intermittency. To analyze these phenomena, we suggest a constructive approach combining the study of the peculiarities of deterministic phase portrait, and stochastic sensitivity of attractors. We show how parametric noise can stabilize the unstable equilibrium and transform dynamics of Kaldor system from order to chaos.

1. Introduction

Dynamic processes in the economy and finance are often very complex and unpredictable. A temporary stabilization is replaced by intervals of abrupt changes, expansions alternate with recessions, periodic fluctuations lose its regularity and start to look like chaotic. An identification of the intrinsic mechanisms that give rise to these changes is a challenging problem of evolutionary economics. An understanding of such mechanisms would not only help to analyze current changes of the main economic parameters, but also to predict their future behavior. An analysis of these mechanisms is possible only on the basis of mathematical modeling of studied dynamic processes.

A first step in the understanding of the internal mechanism of transitions from the equilibrium to oscillatory regimes was carried out by Kaldor [21] who proposed a mathematical approach to explain the periodic changes of the main economic variables of the “income” (Y) and “capital stock” (K). In [11], this idea was formalized in a mathematical model defined by the two-dimensional nonlinear system of differential equations. In this model, the transition from equilibrium to business cycles occurs via Andronov-Hopf bifurcation. Kaldor model was an important example showing how the problems of economic dynamics can be solved within the framework of the qualitative theory of differential equations. This model has clarified that the underlying reason of the appearance of business cycles is not the external periodic action, such as seasonal changes, but the intrinsic properties associated with the nonlinearity of basic economic relationships. Since these early papers, the Kaldor model and its modifications were studied by many researchers [10,12,14,20,22,23,25,29–31]. Along with the Kaldor model, other dynamic models of business cycles, for example, models of Goodwin [17,26] and Pu [28], have been proposed.

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However, in the framework of two-dimensional deterministic dynamic models, only two regimes can be observed, namely stabilization to the equilibrium and transition to strictly periodic oscillations. Therefore, the proposed deterministic model can not explain the great diversity in the observed variations of the basic economic parameters.

It is well known that any economic system operates in random environment. Intuitively, the small-amplitude stochastic fluctuations around a stable equilibrium or a stable cycle are an expected response to the noise in the Kaldor model. Indeed, such reaction is always observed under rather weak noise. However, an increasing noise can result not only in a natural increase of the variance of stochastic oscillations around the deterministic attractors (equilibria and cycles), but also generate regimes, which have no analogues in the initial unforced deterministic model. It is well known that in nonlinear dynamic systems, random disturbances can induce a wide variety of complex, often counter-intuitive, and non-trivial phenomena such as noise-induced transitions [18,24], stochastic bifurcations [1,2], noise-induced chaos [15], stochastic resonance [13], and so on. A propagation of these ideas on the analysis of complex economic processes was started just recently [7–9,16,19,27].

The aim of present paper is to demonstrate a wide variety of dynamic regimes generated by random noise, on the base of the well-known Kaldor model. Along with the discussion of possible stochastic phenomena, constructive methods of their parametric analysis are suggested. It is known that a rigorous mathematical description of stochastic dynamics is given by the Kolmogorov–Fokker–Planck equation. However, even in the simple two-dimensional case (Kaldor model, for instance), it is very difficult to solve it directly. So, asymptotic methods and approximations are actively used. In present paper, we discuss and apply a constructive approach combining stochastic sensitivity functions (SSF) technique [4,5] and confidence domains method. A brief mathematical background of the SSF technique is given in Appendix.

In Section 2, results of the bifurcation analysis of the deterministic Kaldor model are shortly presented. Here, parametric zones corresponding to three different types of dynamics are described. Depending on the parameters, the deterministic system possesses stable equilibrium, or stable cycle, or coexistence of both. In the last case, dynamics depends also on the choice of the initial state.

In Section 3, we demonstrate and analyze a variety of stochastic regimes in Kaldor model forced by the additive random noise. For weak noise, the stochastic system possesses only two regimes: small-amplitude stochastic oscillations (SASO) around the deterministic stable equilibrium, and large-amplitude stochastic oscillations (LASO) near the deterministic stable cycle. Increasing noise can generate transformation from one regime to another.

In our previous paper [8], we considered the effects of the noise in parametric zone where the deterministic Kaldor model has two coexisting attractors (stable equilibrium and limit cycle). Noise-induced transitions between these two attractors which generate mixed-mode stochastic oscillations were studied in details there.

A transformation from SASO to LASO in the monostable zone where the deterministic model exhibits a stable equilibrium only, is demonstrated in the Section 3.1. In Section 3.2, we study inverse transformation from LASO to SASO in the monostability zone where the stable limit cycle is a single attractor. Here, the counter-intuitive effect of the generation of SASO near the unstable equilibrium is shown. With further increase of noise, multiple transitions between LASO and SASO, so called mixed-mode stochastic oscillations (MMSO), can occur. In Section 3.3, we demonstrate a phenomenon of noise-induced MMSO in Kaldor model not only in the bistability zone, where such transitions between coexisting deterministic attractors (equilibrium and cycle) are naturally expected, but also in monostability cases with single attractors, where MMSO are unpredictable from the deterministic point of view. In Section 3.4, we consider a stochastic phenomenon of generation of bi-equilibrium stochastic oscillations (BESO). This regime is characterized by the intermittency of SASO near two fixed points, and LASO between them. Here, we demonstrate a possibility of BESO generation in two parametric zones of Kaldor model, namely in monostability zone with single equilibrium, and in bistability zone with coexisting equilibrium and cycle.

Along with the additive noise that simulates the external random disturbances, it is important to take into account random disturbances acting on the key internal parameters of the studied Kaldor model. Such parametric noise can generate new specific stochastic phenomena.

Important examples of the impact of parametric noise are discussed in Section 4. Here, it is shown that in the same deterministic system, the parametric noise can result in the absolutely opposite reaction. In Section 4.1, we discuss a stabilization of the unstable equilibrium by parametric noise. In Section 4.2, it is demonstrated how parametric noise transforms dynamics of Kaldor system from order to chaos.

2. Regimes of deterministic dynamics

We consider the Kaldor one-sector business model of a closed economy which describes the dynamics of the income Y and the capital stock K in terms of the system of two differential equations [11,21,25]

$$\begin{cases}\dot{Y} = \alpha(I(Y,K) - S(Y,K)),\\ \dot{K} = I(Y,K) - \delta K,\end{cases} \tag{1}$$

where adjustment coefficient $\alpha > 0$ characterizes the speed of reaction of the system to the difference between investments $I(Y, K)$ and savings $S(Y, K)$, and the parameter $\delta \in (0, 1)$ represents the depreciation rate of capital stock.

In present paper, the following representation of the investment function $I(Y, K)$ and saving function $S(Y, K)$ is used:

$$S(Y,K) = \gamma Y, \quad I(Y,K) = I(Y) - \beta K \quad (\gamma > 0,\ \beta > 0).$$