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Research paper

On the periodic Toda lattice hierarchy with an integral source

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This work is devoted to the application of inverse spectral problem for integration of the periodic Toda lattice hierarchy with an integral type source. The effective method is presented of constructing the periodic Toda lattice hierarchy with an integral source.

1. Introduction

The Toda lattice [1] is a simple model for a nonlinear one-dimensional crystal that describes the motion of a chain of particles with exponential interactions of the nearest neighbors. The equation of motion for such a system is given by

$$\frac{d^2u_n}{dt^2} = \exp\left(u_{n-1} - u_n\right) - \exp\left(u_n - u_{n+1}\right), \quad n \in \mathbb{Z},$$

where $u_n(t)$ is the coordinate of the nth atom in a lattice. It is well known that, by means of the Flaschka variables [2], the Toda lattice has the form

$$\begin{cases} \dot{a}_n = a_n(b_{n+1} - b_n), \\ \dot{b}_n = 2(a_n^2 - a_{n-1}^2), \qquad n \in \mathbb{Z}. \end{cases}$$

This equation has different practical applications. For example, the Toda lattice model of DNA in the field of biology [3]. Moreover, one important property of the Toda lattice type equations is the existence of so called soliton solutions. There

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is a close relation between the existence of soliton solutions and the integrability of equations: the known research results show that all the integrable systems have soliton solutions [4].

Integrable nonlinear evolution equations with self-consistent sources have received much attention in the recent research literature. They have important applications in plasma physics, hydrodynamics, solid state physics, etc. [5–11]. For example, the KdV equation, which includes an integral type self-consistent source, was considered in [8]. By this type of equations, the interaction of long and short capillary-gravity waves can be described [9]. Other important soliton equation with a self-consistent source is the nonlinear Schrödinger equation with self-consistent sources which describes the nonlinear interaction of an ion acoustic wave in the two component homogeneous plasma with the electrostatic high frequency wave [10].

Usually, the right-hand side of an integrable equation with a self-consistent source consists of terms multiplied by integral factors depending on all the dynamical variables. Different techniques have been used to construct their solutions, such as inverse scattering [6,7,12–14], Darboux transformation [15–18] or Hirota bi-linear methods [19–21]. Other aspects on integration of nonlinear periodical systems are presented in [22–32].

In [33], Toda lattice hierarchy with self-consistent sources is constructed and studied by means of the Darboux transformation. In [34,35], the authors presented periodic Toda lattice hierarchy without source and showed its integrability by using inverse spectral method of the discrete Hill equation.

In this work, the new method is presented of constructing the N-periodic Toda lattice hierarchy with integral-type self-consistent sources.

The purpose of this paper is to derive representations for the solutions of the constructed new system in the framework of the inverse spectral problem for the discrete Hill equation. In the one-gap case, we write the explicit formulas for solutions of the problem under consideration, expressed in terms of the Jacobi elliptic functions.

The considered new system, similarly to [36,37], can be used in some models of special types of electric transmission lines.

The paper is organized as follows. In Section 2, we first review some results on the Toda lattice and formulate the solved problem. We give some basic information about the spectral theory for the discrete Hill equation in Section 3. In Section 4, we present an effective method of solving the inverse spectral problem for the discrete Hill equation which is very comfortable for numerical calculation. Sections 5 and 6 are devoted to construction of the N-periodic Toda lattice hierarchy with integral-type self-consistent sources and description of the evolution of the spectral data corresponding to the problem. In Section 7, we illustrate the application of the main result for the one-gap case.

2. Formulation of the problem

In this section, we present the formulation of the problem which is being considered. In the present paper, we consider N-periodic Toda lattice hierarchy with integral-type self-consistent source

$$\begin{cases} \dot{a}_n = P_m(a_n, b_n) + a_n \int_E \tilde{\theta}_{N+1}(\lambda, t)[\psi^-_{n+1}(\lambda, t)\psi^+_{n+1}(\lambda, t) - \psi^-_n(\lambda, t)\psi^+_n(\lambda, t)]d\lambda, \\ \dot{b}_n = Q_m(a_n, b_n) + a_n \int_E \tilde{\theta}_{N+1}(\lambda, t)[\psi^-_n(\lambda, t)\psi^+_{n+1}(\lambda, t) + \psi^-_{n+1}(\lambda, t)\psi^+_n(\lambda, t)]\,d\lambda \\ \qquad - a_{n-1} \int_E \tilde{\theta}_{N+1}(\lambda, t)[\psi^-_n(\lambda, t)\psi^+_{n-1}(\lambda, t) + \psi^-_{n-1}(\lambda, t)\psi^+_n(\lambda, t)]\,d\lambda, \\ a_{n+N} = a_n, \quad b_{n+N} = b_n, \quad a_n > 0, \quad n \in \mathbb{Z}, \quad t \in \mathbb{R}, \end{cases} \tag{1}$$

and the initial conditions

$$a_n(0) = a_n^0, \quad b_n(0) = b_n^0, \quad n \in \mathbb{Z}, \tag{2}$$

with the given N-periodic sequences a_n^0 and b_n^0, $n \in \mathbb{Z}$, where

$$P_m(a_n, b_n) = a_n[-\beta_{n,m} - \beta_{n+1,m} + b_{n+1}\alpha_{n+1,m}],$$
$$Q_m(a_n, b_n) = a_n^2\alpha_{n+1,m} - a_{n-1}^2\alpha_{n-1,m} - 2b_n\beta_{n,m} + b_n^2\alpha_{n,m}, \quad m \in \mathbb{N}, \quad t \in \mathbb{R},$$

and $\{\alpha_{n,\,s}(t)\}_{0 \le s \le m}$, $\{\beta_{n,\,s}(t)\}_{0 \le s \le m}$ satisfy the recursion relations

$$\alpha_{n,0} = 0, \quad \beta_{n,0} = c_0, \quad \alpha_{n,1} = 2c_0, \quad c_0 = const,$$
$$\beta_{n,s-1} - \beta_{n-1,s-1} = b_n(\beta_{n,s-2} - \beta_{n-1,s-2}) - a_n^2\alpha_{n+1,s-2} + a_{n-1}^2\alpha_{n-1,s-2}, \quad 2 \le s \le m,$$
$$\alpha_{n,s} = b_n\alpha_{n,s-1} - \beta_{n-1,s-1} - \beta_{n,s-1}, \quad 2 \le s \le m,$$
$$\beta_{n,m} = \frac{a_{n-1}^2}{2}\alpha_{n-1,m-1} - \frac{a_n^2}{2}\alpha_{n+1,m-1} + \frac{b_n^2}{2}\alpha_{n,m-1} - b_n\beta_{n-1,m-1}.$$

Varying $m \in \mathbb{N}$, it yields to N-periodic Toda lattice hierarchy with integral-type self-consistent source (1). The function sequences $\{a_n(t)\}_{-\infty}^{\infty}$, $\{b_n(t)\}_{-\infty}^{\infty}$, $\{\psi_n^{\pm}(\lambda, t)\}_{-\infty}^{\infty}$ – are unknown vector-functions, besides $\{\psi_n^{\pm}(\lambda, t)\}_{-\infty}^{\infty}$ are the Floquet-Bloch solutions for the discrete Hill's equation

$$(L(t)y)_n \equiv a_{n-1}y_{n-1} + b_ny_n + a_ny_{n+1} = \lambda y_n \tag{3}$$