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Research paper

A new look on the stabilization of inverted pendulum with parametric excitation and large random frequencies: Analytical and numerical approaches

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ABSTRACT

In this paper we explore the stability of an inverted pendulum with generalized parametric excitation described by a superposition of N sines with different frequencies and phases. We show that when the amplitude is scaled with the frequency we obtain the stabilization of the real inverted pendulum, i.e., with values of g according to planet Earth ($g \approx 9.8$ m/s^2) for high frequencies. By randomly sorting the frequencies, we obtain a critical amplitude in light of perturbative theory in classical mechanics which is numerically tested by exploring its validity regime in many alternatives. We also analyse the effects when different values of N as well as the pendulum size l are taken into account.

1. Introduction

The inverted pendulum and its stability are subjects widely explored in Physics, Engineering, Biology [1], and many other areas due to its technological importance. An inverted pendulum is unstable unless some kind of excitation/vibration is applied to its suspension point (its basis). Kaptiza [2,3] observed that an inverted pendulum should be stabilized by rapidly oscillating its basis. The limit of stability considering a periodic function at the basis of pendulum has been studied by many authors (see for example [4–6]). In another context, chaos and bifurcations have been studied for a sinusoidal excitation where both excitation frequencies and amplitudes were varied [7]. However, this excitation can be more general opening a long way to explore the stochastic aspects in the stabilization.

By denoting $z(t)$ as a vertical excitation, the Lagrangian of a pendulum with mass m can be written as

$$\mathcal{L}(\theta,\dot{\theta},z,\dot{z}) = \frac{1}{2}ml^2\left(\dot{\theta}^2+\frac{1}{l^2}\dot{z}^2(t)\right) - ml\dot{z}(t)\dot{\theta}\sin\theta \\ -mgz(t) - mgl\cos\theta \quad (1)$$

where the axis z is oriented up, $\vec{a} = -g\hat{z}$ is the gravitational acceleration, and l is the pendulum length, which in turn, leads to the following equation of motion

$$\frac{d^2\theta}{dt^2} = \frac{g}{l}\left(1+\frac{1}{g}\ddot{z}\right)\sin\theta. \quad (2)$$

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Assuming $\ddot{z}$ as a generic time-dependent function and taking into account the limit of small oscillations $\sin\theta \approx \theta$, one has

$$\ddot{\theta} = \left(\omega_0^2 + \frac{1}{l}\ddot{z}\right)\theta, \tag{3}$$

where $\omega_0^2 = g/l$.

An interesting choice is to consider a parametric excitation

$$z(t) = \sum_{i=1}^{N} A_i \sin(\omega_i t + \varphi_i), \tag{4}$$

where A_i, ω_i, and φ_i are arbitrary amplitudes, frequencies and phases, respectively, and $i = 1, \ldots, N$. A detailed study of this kind of excitation in an inverted pendulum can be found, for example, in Refs.[1,8]. In a very different context, Dettmann et al. [9] had obtained an equation for the distance between two photons that propagate in a universe of negative curvature. This distance can be written as

$$\ddot{\theta} = (1 - f(t))\theta. \tag{5}$$

At a first glance, this result seems to be a particular case of Eq. (3), where $\omega_0^2 = 1$ and $f(t)$ is a stochastic forcing function that, in their particular case, takes into account the perturbation in the curvature due to mass distribution $\omega_0^2 = 1$. They studied the stochastic stabilization of Eq. (5) by considering

$$f(t) = f_D(t) = A\sum_{i=1}^{N} \sin(\omega_i t + \varphi_i), \tag{6}$$

where A is a control parameter and $\{\omega_i, \varphi_i\}_{i=1}^{N}$ are chosen independently at random according to a uniform distribution defined on supports: $[\omega_{\min}, \omega_{\max}]$ and $[\phi_{\min}, \phi_{\max}]$ respectively.

It is important to emphasize that by simply making $\omega_0^2 = 1$ in Eq. (3) with $A_i = A$ in Eq. (4), we do not recover Eq. (5), since the usual $f(t)$ considered for a regular pendulum is

$$f(t) = f_S(t) = -\frac{1}{l}\ddot{z}(t) = \frac{1}{l}\sum_{i=1}^{N} A_i\omega_i^2 \sin(\omega_i t + \varphi_i). \tag{7}$$

At this point two technical problems occur. Firstly, $\omega_0^2 = 1$ means the specific case of a huge pendulum ($l \approx 9.8$ m). Secondly, we should incorporate the gravity g in A. However, the term ω_i^2 does not exist in the original problem considered by Dettman et al. [9].

The main contributions of this paper is related to the stabilization of the inverted pendulum or similar system. Here, we answer the following two questions and compare both situations:

1. Is it possible to stabilize an inverted pendulum in a general situation, i.e., by considering the Eq. (3) with the parametric excitation $z(t) = A\sum_{i=1}^{N} \sin(\omega_i t + \varphi_i)$ with random frequencies uniformly distributed in $[\omega_{\min}, \omega_{\max}]$? If yes, what should be the parameters ω_0 and A?
2. Is it possible to obtain a more general stabilization criteria by considering any values of parameters $\omega_{\min}, \omega_{\max}$ and A? This question arises because Dettman et al. showed that the problem for a particular cosmological application (Eqs. (5) and (6)) can be "stochastically" stabilized when considering a specific choice of $\omega_{\min}, \omega_{\max}$ and A.

Throughout this work we will present the answer of these questions. However, we would like to point out that the answer is yes, we are able to stabilize the inverted pendulum by considering more general criterias.

Our manuscript is organized as follows: In the next section, we present the perturbative calculations in detail and show a general solution for the problem. For this purpose, we consider the more general equation

$$\ddot{\theta} = \left(\omega_0^2 - f(t)\right)\theta \tag{8}$$

with a more general function

$$f(t) = \sum_{i=1}^{N} A_i^* \sin(\omega_i t + \varphi_i)$$

where $\omega_0^2 = 1$ and $A_i^* = A$ correspond to the cosmological problem (Problem I) and $\omega_0^2 = \frac{g}{l}$ and $A_i^* = \frac{\omega_i^2}{l}A_i$, $i = 1, 2, \ldots, N$ correspond to the general regular inverted pendulum (Problem II). In Section 3 we show our results and the conclusions are presented in Section 3.