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Research paper

Fast multipole method applied to Lagrangian simulations of vortical flows

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ABSTRACT

Lagrangian simulations of unsteady vortical flows are accelerated by the multi-level fast multipole method, FMM. The combination of the FMM algorithm with a discrete vortex method, DVM, is discussed for free domain and periodic problems with focus on implementation details to reduce numerical dissipation and avoid spurious solutions in unsteady inviscid flows. An assessment of the FMM-DVM accuracy is presented through a comparison with the direct calculation of the Biot–Savart law for the simulation of the temporal evolution of an aircraft wake in the Trefftz plane. The role of several parameters such as time step restriction, truncation of the FMM series expansion, number of particles in the wake discretization and machine precision is investigated and we show how to avoid spurious instabilities. The FMM-DVM is also applied to compute the evolution of a temporal shear layer with periodic boundary conditions. A novel approach is proposed to achieve accurate solutions in the periodic FMM. This approach avoids a spurious precession of the periodic shear layer and solutions are shown to converge to the direct Biot–Savart calculation using a cotangent function.

1. Introduction

We solve canonical flows with practical applications in physics and engineering using the discrete vortex method, DVM, which is a Lagrangian method that avoids computational mesh generation. In this method, the Navier–Stokes equations are solved in the vorticity form by the discretization of the vorticity field using N discrete elements, which are transported with the local flow velocity. As shown by Chorin [1], the convection and diffusion processes appearing in the vorticity equation can be solved separately in the DVM and the calculation of the non-linear term is avoided through application of the material derivative, associated to Lagrangian methods. The convection term may be solved using a potential vortex model which does not introduce numerical diffusivity. In this context, spurious numerical instabilities may appear when the DVM is employed for the solution of inviscid flow problems modeled by vortex sheets. In order to regularize the solution of potential vortices, other models may be employed adding numerical diffusion to the vorticity field. Recently, the DVM has been applied to solve two dimensional problems including airfoil leading edge separation [2,3], unsteady motion of a pitching airfoil [4] and cylinder wake instability [5]. A review of the method can be seen in Barba et al. [6] and Aref [7].

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In order to solve the velocity field using the DVM, one must compute the interactions among discrete vortex particles which are governed by the Biot–Savart law. This leads to a computational cost proportional to O(N^2) which may be reduced by the implementation of a fast algorithm. Here, the fast multipole method, FMM, is chosen since it reduces the computational cost from O(N^2) to O(N) when N is sufficiently large and a hierarchical multi-level algorithm is used. The method was initially proposed by Greengard and Rokhlin [8], and later optimized by Carrier et al. [9] for the solution of potential fields. The FMM has been applied in the literature to accelerate the solution of several problems involving potential flows [10], acoustic and electromagnetic scattering [10–13] and aero-acoustics [14,15]. An efficient implementation of the FMM was proposed by Gumerov and Duraiswami for the simulation of vortical flows [16].

Fast algorithms have been also applied in combination with particle methods for the solution of problems with periodic boundary conditions [17,18]. For instance, Yokota and Barba, and Yokota and Obi employed the periodic FMM to study isotropic turbulence and homogeneous shear flows, respectively [19,20]. Sakajo and Okamoto [21] present an approach in which an exponential mapping transforms the periodic DVM function into a rational function. Then, these authors employ a fast method [22] to simulate the Kelvin–Helmholtz instability problem. Recently, Marple et al. [23] developed a fast algorithm to solve multiphase flows and applied the method for the solution of Stokes flows in periodic channels of arbitrary geometry.

In the present work, the coupled FMM-DVM algorithm is employed to simulate the time evolution of an aircraft wake and a periodic shear layer. The first problem represents the formation of wing tip vortices which appear in the wake of an airplane. Aerodynamic wakes of large aircraft and their induced velocity may cause serious hazards to smaller aircraft, affecting the take-off and landing operations in airports. A simplified but representative model of the problem is the evolution of a vortex sheet in the Trefftz plane, as explained in details by Smith [24]. The second problem is relevant in the context of several problems of engineering and physics involving ocean mixing, cloud formation, multi-phase flow, combustion and jet flows (Murray et al. [25], Smyth and Moum [26], Herrmann [27]). In viscous flow problems, physical dissipation will overcome the numerical dissipation that is introduced in the computation of the convective step via time marching scheme and vortex regularization [28]. However, in the study of hydrodynamic instabilities which may appear in inviscid flows, numerical dissipation can be an issue and accurate solutions are required.

This study focus on the implementation details of the FMM-DVM algorithm with the aim to reduce numerical dissipation in unsteady inviscid flows. We evaluate the pros and cons from coupling both the FMM and DVM in order to avoid spurious solutions. To do so, we analyze the role of temporal discretization and aspects of regularization of vortex models. We also provide an assessment of the parameters which control the errors of the FMM, for instance, the refinement level and the number of terms in the series expansions. Discretization effects are discussed for the solution of the Trefftz plane problem showing that a convergence to the truly inviscid solution, without instabilities, can be achieved. Previous work has showed the presence of spurious instabilities in the calculation of this problem for quasi-inviscid solutions (Fink and Soh [29], Krasny [30–32], Abid and Verga [33]). Lastly, a periodic problem is solved with a FMM implementation which reduces truncation error using balanced periodic boundary conditions to avoid a spurious precession of the infinite shear layer.

2. Numerical methodology

2.1. Discrete vortex method

The discrete vortex method solves the Navier-Stokes equations in the vorticity form. For an incompressible Newtonian fluid, the 2D vorticity transport equation in non-dimensional variables is given by

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = \frac{1}{Re} \nabla^2 \omega, \tag{1}$$

where $\mathbf{u} = u\,\hat{\imath} + v\,\hat{\jmath}$ is the velocity vector, $\omega = \nabla \times \mathbf{u}$ is the z-vorticity component and Re is the Reynolds number. Chorin [1] proposed a numerical solution of Eq. (1) solving separately the inviscid and viscous terms in two fractional steps. The first step considers the flow to be inviscid, *i.e.*, $Re \to \infty$, leading to

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = \frac{\mathrm{D}\omega}{\mathrm{D}t} = 0, \tag{2}$$

while the second step solves the viscous effects

$$\frac{\partial \omega}{\partial t} = \frac{1}{Re} \nabla^2 \omega. \tag{3}$$

The calculation of the inviscid term requires the solution of the Biot–Savart law, which computes the velocity field due to the vortex interactions. For a two-dimensional problem, the Biot–Savart law is written as

$$\mathbf{u}(\mathbf{x}, t) = \frac{1}{2\pi} \int_{S'} \frac{\boldsymbol{\omega}(\mathbf{x}', t) \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^2} \, dS'. \tag{4}$$