



Research paper

Perturbation theory for the Fokker–Planck operator in chaos

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ABSTRACT

The stationary distribution of a fully chaotic system typically exhibits a fractal structure, which dramatically changes if the dynamical equations are even slightly modified. Perturbative techniques are not expected to work in this situation. In contrast, the presence of additive noise smooths out the stationary distribution, and perturbation theory becomes applicable. We show that a perturbation expansion for the Fokker–Planck evolution operator yields surprisingly accurate estimates of long-time averages in an otherwise unlikely scenario.

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1. Introduction

The divergence of perturbation series due to small denominators in the vicinity of unstable fixed points, observed by Poincaré [1,43], was historically among the first hints of chaotic dynamics. This is enough to tell us that perturbative analysis might not get along with chaos.

Nonetheless, there have been successful attempts to treat instabilities with perturbation techniques, the first tracing back to Shimizu [2–4]. Shimizu model systems have a parameter which gradually drives them from periodicity to the onset of chaos. In that context, fast and slow time scales in the periodic phase allow for closed-form, asymptotic solutions, which shed light on the transition to chaos.

The validity of perturbation theory for systems far from equilibrium is intimately related to the existence of a linear response to the perturbation [5], an issue that has sparked great interest and some controversy [6–8] over the years. For maps in particular, we know that all smooth hyperbolic [9] and some partially hyperbolic diffeomorphisms [10] admit a linear response. The question is instead still open for a variety of physically interesting models, including the Hénon family, as well as piecewise hyperbolic maps [11,12].

The object of our investigation is systems already deep in the fully chaotic regime, where in general one cannot separate time scales. We wish to determine whether the dynamics ever lends itself to a perturbative approach. The issue was notably addressed more than 20 years ago by Ershov [13] in the context of one-dimensional maps, with the result that a perturbation of $O(\epsilon)$ to the original map would bring about a deviation up to $O(\epsilon \log \epsilon)$ in the response (see ref. [14] for a proof). Chaos breaks the proportionality between control parameter and statistical characteristics, disrupting the very basis of perturbation theory. Shortly later Ershov [15] established that the Perron–Frobenius operator becomes continuous in the

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presence of additive, uncorrelated noise. The response to perturbations of a density is then restored to $O(\epsilon)$, allowing in principle for perturbative calculations.

Proportionality between parameters and observables plays a major role in chaotic time series analysis [16], as first pointed out in ref. [17], where perturbations may be fruitfully used to test the robustness of the model [18]. In particular, dealing with long time series often boils down to building a Markov chain from the data. This can be expressed as a discretization of the Perron–Frobenius operator of the dynamics [19], whose leading eigenfunction, or invariant density, is used for the estimation of long time averages [20].

We organize the paper as follows: the *Fokker–Planck evolution operator* for a discrete-time dynamical system is introduced in Section 2. In Section 3 we add a small deterministic correction to a weakly noisy map, and develop a perturbation technique to approximate the corresponding Fokker–Planck operator. Our goal is to accurately estimate observables such as the escape rate from a noisy attractor, by using the stationary distribution (or invariant density, for a closed system) of the unperturbed noisy system. Direct numerical simulations of the weakly-noisy Lozi map with a small correction are compared to the perturbative approach in Section 4. Our main result is that the perturbative estimates are in a close agreement with the outcomes of direct numerical simulations for a range of perturbations about an order of magnitude larger than expected. That constitutes quantitative evidence that the noise restores structural stability of the system, with the response proportional to the noise amplitude. Conclusions and comments are given in Section 5.

2. The Fokker–Planck operator

The problem that we are considering involves adding a random variable to a map in d dimensions that would otherwise exhibit deterministic chaos (the subscript n represents time iteration):

$$x_{n+1} = f(x_n) + \xi_n \quad (1)$$

The random variables ξ_n are uncorrelated in time, independent, and distributed according to a Gaussian with a $[d \times d]$ covariance matrix

$$\Delta(x_n)_{ij} = \langle \xi_n^{(i)} \xi_n^{(j)} \rangle \delta_{ij}, \quad (2)$$

which we allow to vary in position x_n but not in time. Time is referenced by n . i and j range from 1 to d . Here we shall consider the evolution of densities of trajectories, according to the Fokker–Planck picture [21]. In discrete time, a distribution moves one time step according to the operator [22,23]

$$\begin{aligned} \phi_{n+1}(y) &= \mathcal{L} \phi_n(y) = \int \phi_n(x) \exp \left\{ -\frac{1}{2} [y - f(x)]^\top \Delta^{-1}(x) [y - f(x)] \right\} [dx] \\ [dx] &= \frac{d^d x}{|\det(2\pi \Delta(x))|^{1/2}}. \end{aligned} \quad (3)$$

The best information we can get about the long time behavior of the system is statistical. Recalling that we are interested in perturbations of steady states, the expectation value of any observable $a(x)$ can be found by knowing the escape rate γ and stationary distribution $\rho(x)$

$$\langle a \rangle = \int e^{\gamma} \rho(x) a(x) dx. \quad (4)$$

The escape rate and the stationary distribution are respectively the logarithm of the maximal modulus (leading) eigenvalue and the leading eigenfunction of the Fokker–Planck operator,

$$\mathcal{L} \rho(x) = e^{-\gamma} \rho(x). \quad (5)$$

As $\mathcal{L}$ is a non-negative operator, its leading eigenvalue is non-degenerate, real, and positive, and the corresponding eigenvector has non-negative coordinates, by Perron–Frobenius theorem [24,25]. This property is also relevant for the applicability of perturbation theory.

The Fokker–Planck operator and its adjoint

$$\mathcal{L}^\dagger \phi_n(y) = \int \phi_n(x) \exp \left\{ -\frac{1}{2} [x - f(y)]^\top \Delta^{-1}(y) [x - f(y)] \right\} [dx] \quad (6)$$

have a whole spectrum of distinct *right* and *left* eigenfunctions, which contain information on how an initial density decays to the stationary distribution,

$$\mathcal{L} \rho_i(x) = \Gamma_i \rho_i(x) \quad \mathcal{L}^\dagger \tilde{\rho}_i(x) = \Gamma_i^* \tilde{\rho}_i(x). \quad (7)$$

Importantly, Eq. (3) defines an integral operator with a $L_2(R^d)$ kernel (said of the Hilbert–Schmidt class [26]), and as such, it is bounded and thus continuous on its support, that is $\|\rho_\epsilon - \rho\| \rightarrow 0$ implies $\|\mathcal{L} \rho_\epsilon - \mathcal{L} \rho\| \rightarrow 0$ and vice versa. As mentioned in the introduction, that is the main difference with the noiseless Perron–Frobenius operator, and the condition for us to apply perturbation theory (details are given in Appendix B).

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