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Research paper

Quasi-integrability in deformed sine-Gordon models and infinite towers of conserved charges

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ABSTRACT

We have studied the space-reflection symmetries of some soliton solutions of deformed sine-Gordon models in the context of the quasi-integrability concept. Considering a dual pair of anomalous Lax representations of the deformed model we compute analytically and numerically an infinite number of alternating conserved and asymptotically conserved charges through a modification of the usual techniques of integrable field theories. The charges associated to two-solitons with a definite parity under space-reflection symmetry, i.e. kink-kink (odd parity) and kink-antikink (even parity) scatterings with equal and opposite velocities, split into two infinite towers of conserved and asymptotically conserved charges. For two-solitons without definite parity under space-reflection symmetry (kink-kink and kink-antikink scatterings with unequal and opposite velocities) our numerical results show the existence of the asymptotically conserved charges only. However, we show that in the center-of-mass reference frame of the two solitons the parity symmetries and their associated set of exactly conserved charges can be restored. Moreover, the positive parity breather-like (kink-antikink bound state) solution exhibits a tower of exactly conserved charges and a subset of charges which are periodic in time. We back up our results with extensive numerical simulations which also demonstrate the existence of long lived breather-like states in these models. The time evolution has been simulated by the 4th order Runge–Kutta method supplied with non-reflecting boundary conditions.

1. Introduction

Solitons and integrable systems play an important role in the study of non-linear phenomena because often they appear in the description of some physical systems. The soliton properties are intimately related to the integrability of the relevant mathematical models in which they arise [1,2]. Some deformations of these theories have been shown to possess solitary waves that behave in a scattering process in a similar way to true solitons [3–10]. The deformations of the relativistic integrable $SU(N)$ Toda [9] (the $N = 2$ case is the sine-Gordon (SG) model in disguise [3,5,10]) and Bullough–Dodd (BD) [6] models have been shown to possess an infinite number of quantities which are not exactly time-independent but are, however, asymptotically conserved. Similar phenomena have been observed in the deformations of the non-relativistic focusing and defocusing non-linear Schrödinger (NLS) model possessing bright and dark solitons [4,7,8]. For earlier observations on related phenomena, such as the elastic scattering of solitons in some non-integrable theories, see e.g. [11].

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Some calculations in solitary wave collisions have been done in the regimes which are close to either integrable or related to a particular relationship between the parameters of the colliding solitons, e.g. high relative velocity, one of the solitons is significantly larger than the other one, fast thin solitons and slow broad solitons are among the cases considered in the literature (see e.g. [12] and references therein). An analytical approach on inelastic solitary wave interactions for a quartic gKdV equation has been considered showing the absence of a pure 2-soliton solution in a special regime [13].

Recently, certain modified defocusing and focusing NLS models [7,8], with dark and bright soliton solutions, respectively, have been shown to exhibit the new feature of an infinite tower of exactly conserved charges. For the special case of two dark (bright) soliton solutions, where the field components are eigenstates of a space-reflection symmetry, they exhibit an alternating sequence of exactly conserved and asymptotically conserved charges for the scattering process of the solitons. These were the distinguishing new features associated to the non-relativistic deformed defocusing (focusing) NLS with dark (bright) soliton solutions, as compared to the previous relativistic quasi-integrable models [3–6,9,10].

This work is a continuation of [3], in which a deformed sine-Gordon model has been used to introduce the quasi-integrability concept. The main result of our paper is that half of the infinite set of quasi-conserved quantities of the deformed sine-Gordon model of ref. [3] are in fact exactly conserved, provided that some two-soliton configurations are eigenstates (even or odd) of the space-reflection operator. By considering linear combinations of the asymptotically conserved charges of [3] we show, through analytical and numerical methods, that one tower of them becomes *exactly* conserved and the other one remains quasi-conserved after the combination; in this way, we have strengthened the arguments of [3] for two-soliton configurations with definite parity. Notice that we have dealt with deformations of the sine-Gordon integrable relativistic field theory and reproduced analogous results to the ones in the non-relativistic non-linear Schrödinger model, as presented in the recent contributions for defocusing NLS with dark solitons [7] and the focusing NLS with bright solitons [8]. In addition, in these last references the authors have found, through numerical simulations, that the first non-trivial anomaly vanishes even in the cases where the space-time parity and space-parity arguments do not indicate they should vanish. We believe that the results in [7,8] and the ones in the present paper open the way for new investigations on the nature of the quasi-integrability phenomena.

We consider the both anomalous Lax representations of the deformed sine-Gordon model, and show that the composition of the space-reflection parity with a special internal symmetry turns out to be a symmetry relating the both anomalous Lax representations. Our analytical and numerical results indicate that the charges associated to two-solitons with a definite parity under space-reflection symmetry, i.e. kink-kink (odd parity) and kink-antikink (even parity) scatterings with equal and opposite velocities, split into two infinite towers of conserved and asymptotically conserved charges. In the case of the positive parity breather-like (kink-antikink bound state) solution one has a tower of exactly conserved charges and a sequence of charges which oscillate around a fixed value. We also show, through numerical simulations, the existence of long lived breather-like states in these models [10], which in our formulation exhibit a subset of exactly conserved charges.

However, it seems to be that such parity property is a necessary condition in order to have the sequence of the exactly conserved charges in the kink-kink and kink-antikink systems. In fact, as we will show by numerical simulations, there are some soliton-like configurations without this symmetry in laboratory coordinates (two-solitons without definite parity under space-reflection symmetry: kink-kink and kink-antikink scatterings with unequal and opposite velocities) which exhibit asymptotically conserved charges only. However, we show that in the center-of-mass reference frame of the two solitons the parity symmetries are restored, and then their associated set of exactly conserved charges would be constructed.

In addition, to simulate the time dependence of field configurations for computing soliton collisions we used the 4th order Runge–Kutta method supplied with non-reflecting boundary conditions suitable to allow the radiation generated as outgoing waves cross the boundary points $x = \pm L$ freely [14]. Our simulations show that some radiation is produced by the soliton systems and the rate of loss of the energy depends on the initial conditions of the system.

The paper is organized as follows: in Section 2 we discuss the dual set of quasi-zero curvature representations introduced in [3], based on the $sl(2)$ loop algebra, for a real scalar field theory subjected to a generic potential, and the dual sets of infinite number of quasi-conservation laws. We discuss the relationship between the space-time parity and asymptotically conserved charges. In Section 3 we introduce the space-reflection parity symmetry and an order two automorphism of the $sl(2)$ loop algebra relating the both dual sets of quasi-conserved quantities. We discuss the relationships between the space-reflection parity and the exactly conserved charges. A tower of new exactly conserved charges is constructed for each field configuration possessing a definite space-reflection parity. In Section 4 we perform the expansion of the theory (2.1) around the sine-Gordon model in power series on the deformation parameter ϵ, and discuss the interplay between the parity of the solution and its dynamics. In Section 5 we study the space-time and space-reflection symmetries of the kink-antikink, kink-kink and breather solitons of the standard sine-Gordon model. In Section 6 the Lorentz transformation is considered in order to study the Lorentz boost transformation of the anomalies and charges. It is shown the vanishing of the anomalies associated to solitary waves. In Section 7 we present the results of our numerical simulations which allowed us to compute and study various properties of the kink-kink, kink-antikink and a system involving a kink and an antikink bound state (breather). In Section 8 we present some conclusions and discussions. The appendix presents useful ϵ-expansions.

2. The model

We consider Lorentz invariant field theories in $(1+1)$-dimensions with equation of motion

$$\partial^2\varphi + \frac{\partial V(\varphi)}{\partial \varphi} = 0, \tag{2.1}$$