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Research paper

Complexity functions for networks: Dynamical hubs and complexity clusters

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ABSTRACT

A method for studying the behavior of the elements of dynamical networks is introduced. We measure the amount of instability stored at each element according to the value of the mean complexity related to this element. Elements with close values of the mean complexity can be unified into complexity clusters; elements with the smallest values of complexities form dynamical hubs. The effectiveness of the method is manifested by its successive application to networks of coupled Lorenz systems.

1. Introduction

In the last years, scientists pay considerable attention to the study of complex networks, i.e., systems consisting of a large number of interacting subsystems (elements). Such systems can occur in physics, chemistry, biology, neuroscience, sociology, etc. Complex networks can be divided into two classes, namely, static and dynamical ones. In the static networks there is no individual dynamics at elements and couplings, and the geometry (topology) of coupling is fixed [1–4]. Therefore, the behavior of such networks is determined only by the geometry (topology) of coupling. On the other hand, in the dynamical ones either elements and/or the couplings and/or geometry (topology) of coupling can vary with time according to the dynamical law [5–7]. While a theory of static networks is now well developed, and, in particular, many different quantities (such as vertex degree, average distance, shortest path, clustering coefficient, etc) have been introduced and used, no general theory exists for dynamical networks (DN), and only a few characteristics reflecting the features of theirs dynamics were presented.

In this article, we restrict our study to the class of DN where the geometry of coupling is fixed, but each element possesses its own dynamics. We suggest a new characteristic for such a DN, which takes into account both the topology of its coupling and the individual dynamics of its elements. We deal with DN that (for the case of continuous time) is a system of the form

$$\dot{y} = F(y), y = \{y_i\}, y_i \in R^{p_i}, p := \sum p_i, \quad (1)$$

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where the vector-function $F = \{F_i\}$ is defined as follows. There is an oriented graph G containing m vertices $v_1, \dots, v_m$. Given index i, the indices $j_1(i), \dots, j_s(i)$ form the neighborhood of i according to the graph G if for each $j_t(i)$ there exists an edge starting at $v_{j_t(i)}$ and ending at v_i, $t = 1, \dots, s$, and every edge in the graph G ending at v_i coincides with one of the edges starting at $v_{j_1(i)}, \dots, v_{j_s(i)}$. Let $K_i = \{j_1(i), \dots, j_s(i)\}$ be the neighborhood of i in the graph G. System (1) can be rewritten as

$$\dot{y}_i = F_i(\{y_j\}, j \in K_i), i = 1, \dots, m. \quad (2)$$

or

$$\dot{y}_i = F_i(y_{j_1(i)}, \dots, y_{j_s(i)}), i = 1, \dots, m.$$

For the DN with discrete time, the dynamical system is generated by the map H: $R^p \to R^p$, $H(y(n)) = y(n+1)$, where $H = \{H_i\}$ and

$$y_i(n+1) = H_i(\{y_j(n)\}, j \in K_i), i = 1, \dots, m. \quad (3)$$

In Section 2, we start with the notions of complexity functions. Then in Section 3 we adjust the notion of a local complexity function to DN and introduce a new characteristic – the so-called mean complexity that determines the “amount of instability” stored at each element of the DN. In Section 4, we present the results of calculations of the mean complexities for a specific DN. We show that a new characteristic allows one to detect dynamical hubs and complexity clusters. Section 5 is devoted to concluding remarks.

2. Local complexity functions: general definitions

In this Section, we recall the definitions of complexity functions and local complexity functions [8] (see [9–12]). We consider a dynamical system (f^t, M), where M is a metric space endowed with a metric $d(x, y)$ and f^t: $M \to M$ is a semi-group of evolution operators (the time $t \geq 0$ can be continuous or discrete). The following definitions are based on the notion of Kolmogorov ε-separability that was adjusted to the dynamical systems area by R. Bowen (see for instance [13]).

Definition 1.

(i) The points $x, y \in M$ are (ε, t)-separated if there exists τ, $0 \leq \tau \leq t$, such that $\mathrm{dist}(f^\tau x, f^\tau y) \geq \varepsilon$.
(ii) A set $A \subset M$ is (ε, t)-separated if every pair $x, y \in A$ is (ε, t)-separated.
(iii) The number

$$C_{\varepsilon,t}(A) = \max\{\text{card } B, B \subset A, B \text{ is } (\varepsilon, t)\text{-separated}\}$$

is called the (ε, t)-complexity of the set A, where card B is the cardinality (the number of elements) of the set B.

The asymptotic behavior (as $t \to \infty$ or/and $\varepsilon \to 0$) of $C_{\varepsilon,t}(A)$ tells us both about instability of the trajectories started at the initial points belonging to A and about metric features of A. Indeed, one can use the topological entropy (see [14])

$$h = \hbar(f|A) := \lim_{\varepsilon \to 0} \overline{\lim_{t \to \infty}} \frac{\ln C_{\varepsilon,t}}{t}$$

and the fractal dimension of A (see [15])

$$b = b(A) = \lim_{t \to \infty} \overline{\lim_{\varepsilon \to 0}} \frac{\ln C_{\varepsilon,t}}{-\ln \varepsilon}$$

to see that if $0 < h < \infty$ and $0 < b < \infty$, then

$$C_{\varepsilon,t}(A) = e^{ht} \cdot e^{-b} \cdot \overline{C}(\varepsilon, t),$$

where $\frac{\ln \overline{C}_{\varepsilon,t}}{t} \to 0$ as $t \to \infty$ and $\frac{\ln \overline{C}_{\varepsilon,t}}{-\ln \varepsilon} \to 0$ as $\varepsilon \to 0$ (see, for instance, [13]), at least for some large values of t and small values of ε. It allows one to distinguish deterministic processes (the case $h < \infty$ and $b < \infty$) from the random ones. Moreover, even for finite values of time the quantity $C_{\varepsilon,t}(A)$ measures the amount of instability developed in the system restricted to a set A of initial conditions in the temporal interval $[0, t]$.

Unfortunately, it is very difficult to work with Definition 1 in specific cases. We shall use a simplified version of it – the so-called local complexity function [8,9,11]. It is based on measuring the divergence of trajectories in the neighborhood of a fixed one.

Definition 2. Given $\varepsilon > 0$, point x_0 and set $A \ni x_0$ with $\mathrm{diam} A \ll \varepsilon$, the set $Q_N = \{x_k\}_{k=1}^N \subset A$ is said to be locally (ε, t)-separated if

(i) for each x_k, there is $\tau_k \in [0, t)$ such that

$$\mathrm{dist}(f^{\tau_k} x_0, f^{\tau_k} x_k) \geq \varepsilon \quad (4)$$

and

$$\mathrm{dist}(f^\tau x_0, f^\tau x_k) < \varepsilon, 0 \leq \tau < \tau_k; \quad (5)$$