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Research paper

Computation of non-monotonic Lyapunov functions for continuous-time systems

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ABSTRACT

In this paper, we propose two methods to compute non-monotonic Lyapunov functions for continuous-time systems which are asymptotically stable. The first method is to solve a linear optimization problem on a compact and bounded set. The proposed linear programming based algorithm delivers a CPA[1] non-monotonic Lyapunov function on a suitable triangulation covering the given compact and bounded set excluding a small neighbourhood of the equilibrium. It is shown that for every asymptotically stable system there exists a suitable triangulation such that the proposed algorithm terminates successfully. The second method is to verify a CPA function constructed based on the values of the norm of the state at all vertices of a suitable triangulation covering the given compact and bounded set is a non-monotonic Lyapunov function on the given set without a small neighbourhood of the equilibrium. It is further proved that if system is asymptotically stable then there exists a suitable triangulation such that the second way works. The comparison of the proposed two methods are discussed via three examples.

1. Introduction

Lyapunov second method proposed by Lyapunov [17] in 1892 plays an important role in stability analysis. Given a dynamic system, if we have a Lyapunov function at hand, then stability of the system can be analysed without calculating the explicit solution of the corresponding ordinary differential (difference) equation. Furthermore, the domain of attraction can be estimated via the obtained Lyapunov function. Therefore, many researchers are very interested in construction of Lyapunov functions.

In order to make it easier to find Lyapunov functions, researchers have been exploring Lyapunov functions with relaxed constraints. In [1], the authors proposed a new sufficient condition for asymptotic stability of time-varying system via a Lyapunov function whom the time derivative of along the trajectory of the state may have negative and positive values. Such a Lyapunov function with relaxed constraints in [1] is later called a non-monotonic Lyapunov function which is allowed to decrease along the system trajectories after a finite number of time steps, and not at every time step. This result was generalised to system with perturbations in [10] by a Lyapunov function with non sign-definite derivative. In [13], based on non-monotonic Lyapunov functions the authors proposed sufficient and necessary conditions for asymptotic stability of discrete-time homogeneous dynamic. The results were extended to discrete-time systems in [4,5]. In [4,5], an alternative

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[1] continuous and piecewise affine

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converse Lyapunov theorem for discrete-time systems was presented based on a non-monotonic Lyapunov function named as a finite step Lyapunov function by the authors. Furthermore, a relaxation of the input-to-state stability Lyapunov function was also discussed in [4,6]. The new converse Lyapunov theorem was extended to continuous-time systems in [3]. Moreover, a method to compute non-monotonic Lyapunov functions via a Massera-type construction was proposed in [3]. Construction of CPA Lyapunov functions via finite-time converse was discussed in [20]. Inspired by results of Geiselhart et al. [5] and Doban and Lazar [3], we are interested in studying non-monotonic Lyapunov functions for continuous-time systems. Some interesting results (Theorem 11, Lemma 16, Remark 17) are obtained. Furthermore, we investigate how to compute non-monotonic Lyapunov functions for continuous-time systems. Based on the computed Lyapunov functions, if the requirements of Lemma 16 are satisfied, then an estimate of the domain of attraction is obtained.

Hafstein in [9] proposed a CPA method to compute Lyapunov functions for continuous-time systems. This method is based on linear programming. The computed Lyapunov function is linearly affine on each simplex. Such a Lyapunov function is calculated via solving a linear optimization problem. This method was extended to computation of Lyapunov functions for differential inclusions in [2], for discrete-time systems in [7], and computation of input-to-state stability Lyapunov functions for systems in [15]. Since the interpolation errors are incorporated in this method, the Lyapunov function delivered by the corresponding algorithm is a true Lyapunov function rather than a numerical approximation. This method is not restrict to computation of Lyapunov functions on a small domain. Therefore, in this paper we investigate how to adapt the CPA method to computation of non-monotonic Lyapunov functions for continuous-time systems. A linear programming based algorithm will be designed for computation of non-monotonic Lyapunov functions. Furthermore, we will describe how to construct non-monotonic Lyapunov functions based on the values of the norm of the state at all vertices of a suitable triangulation covering the given compact and bounded set.

This paper is organised as follows: in Section 2, the notations and preliminaries are introduced. The concept of non-monotonic Lyapunov function is described in Section 3. Furthermore, we discuss the existence of a non-monotonic Lyapunov function with the fixed time steps on a bounded set excluding a small neighbourhood of the equilibrium if system is asymptotically stable. In order to derive asymptotic stability of the considered system in a bounded and compact set by the computed non-monotonic Lyapunov function, the conditions imposed in Lemma 16 are discussed. In Section 4, we propose two ways to compute non-monotonic Lyapunov functions. In Section 4.2, we describe the linear programming based algorithm for computation of non-monotonic Lyapunov functions for continuous-time systems. In Section 4.2.2, we prove that upon successful termination the algorithm yields a non-monotonic Lyapunov function outside a small neighbourhood of the equilibrium, and that successful termination is guaranteed if the system admits a C^1 Lyapunov function with bounded gradient and the suitable triangulation is chosen appropriately. In Section 4.3, we discuss the conditions under which a CPA function constructed based on the values of the norm of the state at all vertices of a suitable triangulation covering the given compact and bounded set is a non-monotonic Lyapunov function outside a small neighbourhood of the equilibrium. Furthermore, it is proved that if system is asymptotically stable then there exists a suitable triangulation such that the constructed CPA function is a non-monotonic Lyapunov function. Three examples are presented to demonstrate how our proposed methods work in Section 5. The comparison of the proposed ways are described in Section 5. Some concluding remarks are discussed in Section 6.

2. Notations and preliminaries

Let $\mathbb{Z}_+,\ \mathbb{R}_+$ denote the nonnegative integers, the nonnegative real numbers, respectively. Given a vector $x\in\mathbb{R}^n$, let x^T represent its transpose. The standard inner product of $x,\ y\in\mathbb{R}^n$ is noted by $\langle x, y\rangle$. For a set $\Omega\subset\mathbb{R}^n$, the boundary, the closure and the complement of Ω are denoted by $\partial\Omega$, $\overline{\Omega}$ and Ω^C respectively. For a vector $x\in\mathbb{R}^n$, let $|x|_1$, $|x|$ and $|x|_\infty=\max\{|x_i|,\ i=1,\dots,n\}$ denote 1-norm, Euclidean norm and ∞-norm respectively. Given a positive constant $r\in\mathbb{R}_+$ and a vector $x_0\in\mathbb{R}^n$, $B(x_0,r)=\{x\in\mathbb{R}^n\,|\,|x-x_0|<r\}$ denotes the open ball of radius r around x_0 in the norm of $|\cdot|$. The induced matrix norm is defined by $|A|=\max_{|x|=1}|Ax|$.

We recall comparison functions which are widely used in stability analysis. A continuous function $\alpha:\mathbb{R}_+\to\mathbb{R}_+$ is called *positive definite* if it satisfies $\alpha(0)=0$ and $\alpha(s)>0$ for all $s>0$. A positive definite function is of *class* $\mathcal{K}$ if it is strictly increasing and of *class* $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. A continuous function $\gamma:\mathbb{R}_+\to\mathbb{R}_+$ is of *class* $\mathcal{L}$ if $\gamma(r)$ is strictly decreasing to 0 as $r\to\infty$ and we call a continuous function $\beta:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ of *class* $\mathcal{KL}$ if it is of class $\mathcal{K}_\infty$ in the first argument and of class $\mathcal{L}$ in the second argument. More details of comparison functions are discussed in [11].

In this paper, we consider system described by

$$\dot{x}=f(x), \tag{1}$$

where $x\in\mathbb{R}^n$, $f:\mathbb{R}^n\mapsto\mathbb{R}^n$ is locally Lipschitz continuous and $f(0)=0$. The solution of (1) is denoted by $x(t,x_0)$ corresponding to an initial condition x_0.

In this paper, we let a bounded and compact set $\Omega\subset\mathbb{R}^n$ satisfy $0\in\operatorname{int}\Omega$, and a real number $L>0$ note the constant such that

$$|f(x)-f(x_0)|\le L|x-x_0|,\ \text{ for } x,\ x_0\in\Omega. \tag{2}$$

Definition 1. The origin of system (1) is stable if for each $\epsilon>0$ there exists a constant $\delta>0$ such that

$$|x(t,x_0)|\le\epsilon \tag{3}$$