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Mathematical modelling of fractional order circuit elements and bioimpedance applications

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A B S T R A C T

In this work a classical derivation of fractional order circuits models is presented. Generalised constitutive equations in terms of fractional Riemann–Liouville derivatives are introduced in the Maxwell's equations for each circuit element. Next the Kirchhoff voltage law is applied in a RCL circuit configuration. It is shown that from basic properties of Fractional Calculus, a fractional differential equation model with Caputo derivatives is obtained. Thus standard initial conditions apply. Finally, models for bioimpedance are revisited.

1. Introduction

In recent decades there has been a great interest on Fractional Calculus and its applications. A historical review of applications is presented in Machado et al. [8].

Models involving fractional derivatives and operators have been found to better describe some real phenomena. In this work our interest is on fractional circuits models. Research on the topic is very active and lively. From the perspective of Systems Theory, an inviting introduction to Fractional Systems is presented in Ortigueira [3]. See also the work of Radwan and Salama [13]. A recent study on fractional circuits elements is that of Machado [9], where a case is made for elements even with complex valued derivatives. Focusing on the applications to fractional order circuits is the work of Elwakil [5]. Therein a motivation for fractional order circuits from Biochemistry and Medicine is described. Also, cardiac electrode tissue impedance and respiratory impedance with fractional circuit models are addressed respectively in Magin [10] and Ionescu, Derom and Keyser [7]. Their discussion on bioimpedance and listed references provide a rather complete and documented overview of the subject.

Of particular interest to this work, is the survey on models from Biology and Biomedicine for fitting impedance data presented in Freeborn [4]. With regards to the Cole Model, the Laplace transform model of a fractional order circuit, it is pointed out that: *"while this model is effective at representing experimentally collected bioimpedance data, it does not provide an explanation of the underlying mechanisms".*

Our objective is to provide some insight on this issue, namely, an explanation for fractional order circuits as models of electromagnetic phenomena in lossy media. Following a classical modelling approach, we derive fractional order circuits models where fractional order derivatives are physically sound. As a first step, we derive models for fractional circuits

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elements by considering generalised constitutive equations in terms of fractional time derivatives in the sense of Riemann–Liouville. These equations are used to relate the electric flux density, the magnetic field intensity and the electric current density with the electric and magnetic fields. For instance, in the fractional resistor we consider a generalised Ohm's law consistent with the empirical Curie's law. The latter is the point of departure of our work, it is discussed in Westerlund and Ekstam [14]. Next we consider the RCL circuit configuration and apply the Maxwell's equations as customary. We are led to a fractional ordinary differential equation in the Riemann–Liouville sense, modelling fractional order circuits. It is well known that initial conditions need not be well defined in the Riemann–Liouville case. We show that it is straightforward to obtain an equivalent model with derivatives in the Caputo sense. Consequently, initial conditions can be prescribed as usual. Finally we specialised to bioimpedance circuit models providing an answer to the query above.

The outline is as follows.

In Section 2 we list some basic definitions and properties of Fractional Calculus, these shall be used freely throughout. In Section 3 we review Curie's law and recall the Maxwell's equations in continuous media. From Curie's law we make a case for a generalised Ohm's law in lossy media. The classical modelling of circuit elements is carried out in Section 4. Therein, generalised laws for the electric flux density and magnetic field intensity are introduced leading to fractional versions of circuit componentes. In Section 5 we apply the Kirchhofff's law of voltages to obtain the desired fractional order circuit model. In Section 6 we focus on bioimpedance circuits models.

2. Preliminaries on fractional calculus

For later reference let us recall some basic properties of Fractional Calculus, see Diethelm [2]. The technical details for the derivation of the models inSections 4 and 5 rely heavily on this material.

Let $s > 0$ and $f : \mathbb{R} \to \mathbb{R}$. The Riemann–Liouville fractional integral of order s centered at 0, $J^s f$ is given by:

$$J^s f(t) = \frac{1}{\Gamma(s)} \int_0^t (t-\sigma)^{s-1} f(\sigma) d\sigma. \tag{1}$$

Hereafter, let $0 < \alpha \leq 1$. The Riemann–Liouville derivative $D^\alpha f(x)$ of order α centered at 0 is defined by

$$D^\alpha f = DJ^{1-\alpha} f. \tag{2}$$

An alternative definition is given by Caputo, $\hat{D}^\alpha f$, namely

$$\hat{D}^\alpha f = J^{1-\alpha} Df.$$

Let $\mathbb{R}^+ = \{t \in \mathbb{R} : t > 0\}$ and $\mathbb{R}_0^+ = \{t \in \mathbb{R} : t \geq 0\}$. Let us denote by $C^0(\mathbb{R}^+)$ and $C^0(\mathbb{R}_0^+)$ spaces of continuous functions Also let $L^1_{loc}(\mathbb{R}^+)$ be the space of locally integrable functions.

The following are well known facts about derivatives and integrals in the sense of Riemann–Liouville:

- $J^r J^s = J^{r+s}, \quad r, s > 0.$ (semigroup property)
- For $\lambda > -1$

  $$D^\alpha t^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} t^{\lambda-\alpha},$$

  giving in particular $D^\alpha t^{\alpha-1} = 0$.
- $D^\alpha J^\alpha f = f \quad$ for all $f \in C^0(\mathbb{R}^+) \cap L^1_{loc}(\mathbb{R}^+)$.
- If $u \in C^0(\mathbb{R}^+) \cap L^1_{loc}(\mathbb{R}^+)$, then the fractional differential equation

  $$D^\alpha u = 0$$

  has $u = ct^{\alpha-1}$, $c \in \mathbb{R}$, as unique solutions.
- If $f, D^\alpha f \in C^0(\mathbb{R}^+) \cap L^1_{loc}(\mathbb{R}^+)$ then

  $$J^\alpha D^\alpha f(t) = f(t) + ct^{\alpha-1}$$

  for some $c \in \mathbb{R}$. When $f \in C^0(\mathbb{R}_0^+)$, $c = 0$.

Our aim is to stress the ideas for the derivation of a fractional circuit model. Thus, we shall assume that the functions under consideration are sufficiently regular. It will become apparent that, with some technicalities, the arguments that follow are valid for functions with weaker properties.

For instance, consider the space $C_r^0(\mathbb{R}^+)$, $r \geq 0$ of functions $f \in C^0(\mathbb{R}^+)$ such that $t^r f \in C^0(\mathbb{R}_0^+)$. Let $r < 1 - s$ and $f \in C_r^0(\mathbb{R}_0^+)$ with $D^s f \in C^0(\mathbb{R}^+) \cap L^1_{loc}(\mathbb{R}^+)$. Then $J^s D^s f = f$.

Finally, we list some properties for the Caputo derivative:

- $\hat{D} f = Df$
- If $f(0) = 0$ then $\hat{D}^\alpha f = D^\alpha f$ and $\hat{D}\hat{D}^\alpha f = \hat{D}^\alpha \hat{D} f = \hat{D}^{1+\alpha} f$