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Research paper

Probabilistic estimates of the maximum norm of random Neumann Fourier series

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ABSTRACT

We study the maximum norm behavior of L^2-normalized random Fourier cosine series with a prescribed large wave number. Precise bounds of this type are an important technical tool in estimates for spinodal decomposition, the celebrated phase separation phenomenon in metal alloys. We derive rigorous asymptotic results as the wave number converges to infinity, and shed light on the behavior of the maximum norm for medium range wave numbers through numerical simulations. Finally, we develop a simplified model for describing the magnitude of extremal values of random Neumann Fourier series. The model describes key features of the development of maxima and can be used to predict them. This is achieved by decoupling magnitude and sign distribution, where the latter plays an important role for the study of the size of the maximum norm. Since we are considering series with Neumann boundary conditions, particular care has to be placed on understanding the behavior of the random sums at the boundary.

1. Introduction

Random series of functions play a significant role in many branches of mathematics and have been studied extensively. See for example the classical references by Adler and Taylor [1] or Kahane [13] and the references therein. Of particular interest in a number of applications is the problem of estimating the maximum norm of random series. For example, in quantum chaos applications the maximum norm of the eigenfunctions of the Laplacian on bounded domains is a measure for localization effects, and it was shown in [2] that one can estimate these norms of deterministic eigenfunctions through random superpositions of plane waves and using methods due to Kahane [13].

Without being exhaustive, other references treating the maximum of a random Fourier series are the book of Marcus and Pisier [18] or the paper by Orsingher [20]. The main difference of these classical references to the results presented here is first, that we consider L^2-normalized Fourier series. Secondly, we cannot rely on simple asymptotic convergence results, as our Fourier series are finite, but the whole range of wave-numbers, over which the Fourier series is summed, is growing but also moving towards infinity in the asymptotic limit.

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The interplay between stochastic techniques and deterministic applications can also be seen in other contexts. Consider for example one of the standard models for phase separation in binary alloys which is due to Cahn and Hilliard [6,7]. They proposed the fourth-order parabolic partial differential equation

$$\partial_t u = -\Delta(\varepsilon^2 \Delta u + h(u)) \qquad \text{in } G\,, \tag{1}$$

subject to homogeneous Neumann boundary conditions $\partial_\nu u = \partial_\nu \Delta u = 0$ on ∂G, and for some sufficiently smooth domain $G \subset \mathbb{R}^d$. In this model, the unknown function u is an order parameter which represents the concentration difference of the two alloy components, i.e., values of u close to $\pm$ 1 represent the pure components, while values in between correspond to mixtures, with $u = 0$ implying equal concentrations of both components. Moreover, the small parameter $\varepsilon > 0$ is a measure for interaction length which is usually on an atomistic length scale, and the nonlinearity is the negative derivative of a double-well potential. A typical example is $h(u) = u - u^3$, while the h in original work of Cahn and Hilliard had logarithmic poles.

If one observes the evolution of the Cahn–Hilliard model originating at some almost constant homogeneous state $u(0, \cdot) \approx m$, and if the initial concentration difference m satisfies the condition $h'(m) > 0$, then it is well-known that provided sufficiently small $\epsilon > 0$ (1) exhibits spontaneous phase separation through a process called spinodal decomposition. The resulting dynamics of the phase variable u exhibits the formation of complicated and intricate patterns, which are generated by the local convergence of the function values of u to $\pm$ 1, while at the same time keeping the number of separating interfaces as small as possible, see for example [4] and the references therein. From a mathematical point of view, spinodal decomposition in the classical Cahn–Hilliard model (1) has been studied in a series of papers [16,17,22,23] through deterministic methods, and they provide an explanation for both the observed complicated patterns and their generation. In particular, it is shown that the Cahn–Hilliard equation exhibits surprising linear behavior even far from the constant stationary state $u \equiv m$, and [22,23] provide lower bounds for the region of linear behavior. Unfortunately, however, these lower bounds turn out to be suboptimal.

It was shown in [24] that optimal lower bounds can be obtained, if instead of the deterministic estimates used in [22,23] one employs a probabilistic approach. More precisely, the suboptimality of the deterministic results is due to the possibility of large ratios between the maximum norm and the $L^2(G)$-norm of functions representing spinodally decomposed patterns, since the deterministic approach needs to incorporate the value of these ratios for all possible patterns. In practice, however, the ratios are reasonably small, and by studying random Fourier series in combination with randomly chosen initial conditions for the deterministic problem (1) one can show that for "typical" initial conditions linear behavior prevails up to much larger distances from the homogeneous state. For more details, see [12,24].

As a model, the deterministic Cahn–Hilliard Eq. (1) ignores thermal fluctuations which are present in any material. This can be resolved by adding a stochastic additive term, see for example [8,15], and leads to the stochastic Cahn–Hilliard–Cook model

$$\partial_t u = -\Delta(\varepsilon^2 \Delta u + h(u)) + \partial_t W \qquad \text{in } G\,, \tag{2}$$

which is again considered subject to Neumann boundary conditions $\partial_\nu u = \partial_\nu \Delta u = 0$ on ∂G, and for some sufficiently smooth domain $G \subset \mathbb{R}^d$. While the nonlinearity h and the interaction parameter ε are as before, the additive noise term $\partial_t W$ is the derivative of a small Q-Wiener process W, which will be described in more detail below. Ideally, one would expect space-time white noise with a small noise strength, which is on the order of an atomistic length scale, too. For a survey of the phase separation dynamics of the Cahn–Hilliard–Cook model (2) see for example [4].

Spinodal decomposition can also be observed in the stochastic Cahn–Hilliard model, and some of the above-mentioned results could be extended to the case of (2). More precisely, in [3] it was shown that results analogous to [16,17] hold, while [5] generalizes the approach of [22,23]. We would like to stress that even though the basic explanation of spinodal decomposition as a phenomenon driven by unexpectedly linear behavior remains, the proof techniques used in the stochastic setting are completely different.

Despite the above results, a complete description of spinodal decomposition which generalizes the approach described in [24] to the stochastic case remains elusive, and we now describe this somewhat surprising fact in more detail. During spinodal decomposition, an initially flat surface $u \approx m$ separates and closely follows the linearized dynamics for unexpectedly large times. In the stochastic setting, the linearized dynamics near the constant solution $u \equiv m$ is described by the evolution equation

$$\partial_t u = Au + \partial_t W \qquad \text{in } G\,, \tag{3}$$

where the linearized operator is given by $A = -\varepsilon^2 \Delta^2 - h'(m)\Delta$, subject to Neumann boundary conditions and average mass zero. Since m is constant, this operator is self-adjoint and has a complete orthonormal system of eigenfunctions $e_k \in L^2(G)$, for $k \in \mathbb{N}$, with associated eigenvalues

$$\lambda_k = \mu_k\big(h'(m) - \varepsilon^2 \mu_k\big) \qquad \text{for} \qquad k \in \mathbb{N}\,. \tag{4}$$

One can easily see that the eigenfunctions e_k are the eigenfunctions of the negative Laplacian subject to homogeneous Neumann boundary conditions, with corresponding ordered eigenvalues $0 < \mu_1 \le \mu_2 \le \ldots \to \infty$. It is well known [11] that the solution of (3) starting at zero is the stochastic convolution

$$W_A(t) = \int_0^t e^{(t-s)A} \mathrm{d}W(s) = \sum_{k\in\mathbb{N}} \alpha_k \int_0^t e^{(t-s)\lambda_k} \mathrm{d}B_k(s) \cdot e_k\,, \tag{5}$$