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Research paper

Limit cycles in planar continuous piecewise linear systems

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ABSTRACT

In this paper an asymmetric planar continuous piecewise linear differential system with three zones $\dot{x} = y - F(x)$, $\dot{y} = -g(x)$ is considered. The aim of this paper gives a completely study of limit cycles when this system satisfies such conditions and the uniqueness equilibrium does not lie in the central region. When $(x - x_0)g(x) > 0$ for $\forall x \neq x_0$ and $y = F(x)$ is a Z-shaped curve, it owns at most two limit cycles, which exist between a linear Hopf bifurcation surface and a double limit cycle bifurcation surface. Moreover, we prove the conjectures proposed by Ponce et al. [27]. When the uniqueness equilibrium lies in the central region, this system has exactly one limit cycles by others. Finally, some numerical examples are demonstrated.

1. Introduction

For some decades, the piecewise linear systems (PWL systems, for short) offer a good platform for the studying on nonlinear dynamical systems, see Refs. [2,5–8,11–17,19,20], and the references therein.

Here, we naturally have a question why PWL systems is so important. A main reason is that the class of PWL systems is a very important class of nonlinear dynamical systems. Firstly, in engineering, many electronic engineering and nonlinear control systems can be accurately modeled by PWL systems. Secondly, in mathematical biology, the PWL systems can constitute approximate models, see [9,21,22]. Thirdly, the discontinuous limit of a smooth oscillator can be changed into piecewise linear systems, see [4]. Besides, the global dynamics of some smooth models can be approximated by PWL systems since nonsmooth piecewise linear phenomena can be thought as the uniform limit of smooth nonlinearity, see [3].

Interestingly, as said in [20], for almost all the instance of dynamics (such as limit cycles, homoclinic loop, heteroclinic loop and strangle attractors ...) found in general smooth nonlinear systems, PWL systems also have similar dynamical behavior. Moreover, PWL systems even have special dynamical behavior, such as grazing bifurcation and sliding bifurcation, see [2].

However, the description of all possible nonlinear responses for PWL systems and their rigorous mathematical justification are tasks only partial undertaken, since many results coming from standard differential dynamics cannot be applied to PWL systems. Moreover, new specific results are still needed even in the seemingly simpler cases. On the other hand, the complete dynamical analysis of PWL systems will be a formidable task since the standard forms of this systems have many parameters.

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Originally released in sixty years of the last century, Andronov et al. started the pioneering investigations of PWL systems in a rigorous, see [1]. In recent years, this is worth mentioning that some Spanish mathematicians (such as Freire, Llibre and Ponce) have many contributions about PWL systems. Undoubtedly, when the work about PWL chaotic systems occurred, the analysis of PWL systems received more attention than before, see [18] and reference therein.

The boundaries between two different linear zones of PWL systems are responsible for the nonsmoothness of the vector field. Moreover, they are assumed to two parallel straight lines in the case of three zones. In applications which deal with some simple models about electronic or nonlinear mechanics, three zones is enough to capture the corresponding dynamics, see [1,2,5]. In particular, as far as we know, the majority of known results for PWL systems with three zones have symmetry, except [15,16,20]. In this paper we focus the attention on limit cycles of non-symmetric PWL systems. It is worth noting that there is few result about at most two limit cycles of non-symmetric PWL systems. We will prove a class PWL systems has at most two limit cycles and exact two ones for special parameter values.

The rest of the paper is organized as follows. In the Section 2, we give main results which are necessary and sufficient conditions of the number of limit cycles for a class of PWL systems. In the Section 3, when the unique equilibrium lies in a straight line, we completely study the number of limit cycles. In the Sections 4 and 5, when the unique equilibrium lies in the right zone, we also completely study the number of limit cycles. Finally, some numerical examples are given in the Section 6.

2. Statement of the main results

We define a asymmetric piecewise linear differential system with three zones defined by

$$\dot{x} = y - F(x), \quad \dot{y} = \delta - g(x), \tag{1}$$

where

$$F(x) = \begin{cases} t_R(x-1) + t_C, & \text{if } x > 1, \\ t_C x, & \text{if } |x| \le 1, \\ t_L(x+1) - t_C, & \text{if } x < -1, \end{cases}$$

and

$$g(x) = \begin{cases} r(x-1) + c, & \text{if } x > 1, \\ cx, & \text{if } |x| \le 1, \\ l(x+1) - c, & \text{if } x < -1. \end{cases}$$

System (1) are introduced in [15,16,20]. Moreover, it has many applications, such as application to the study of a simple oscillator with one memristor(see [16]) and Wien bridge oscillator(see [20]). When the system (1) is symmetric about the origin, i.e.,

$$\delta = 0, \ t_R = t_L, r = l,$$

limit cycles have been studied in Refs. [7,8,11,17]. When the system (1) has two linearity zones, i.e.,

$$c = l, \quad t_C = t_L \text{ or } c = r, \ t_C = t_R,$$

limit cycles have been studied in Refs. [15,16,20]. When the system (1) is asymmetry and has a unique equilibrium, limit cycles of this system have been studied in Refs. [15,16,20].

When l, c, $r > 0$, the system (1) has a unique equilibrium E: (x_0, y_0), where

$$(x_0, y_0) = \begin{cases} \left(\dfrac{\delta - c}{r} + 1, \dfrac{t_R(\delta - c)}{r} + t_C\right), & \text{for } \delta > c; \\ \left(\dfrac{\delta}{c}, \dfrac{t_C\delta}{c}\right), & \text{for } -c \le \delta \le c; \\ \left(\dfrac{\delta + c}{l} - 1, \dfrac{t_L(\delta + c)}{l} - t_C\right), & \text{for } \delta < -c. \end{cases}$$

When $\delta = c$ (resp. $-c$), E lies in the straight lines $x = 1$ (resp. -1); when $\delta > c$(resp. $< -c$), E lies in the right (resp. left) region; when $-c < \delta < c$, E lies in the central region, see [20]. Let

$$e := \frac{\delta - c}{r}.$$

Translating E to the origin, i.e.,

$$(x, y) \to (x + e + 1, \ y + t_R e + t_C),$$

the system (1) can be rewritten as

$$\dot{x} = y - \hat{F}(x), \quad \dot{y} = -\hat{g}(x), \tag{2}$$