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Research paper

New periodic orbits in the planar equal-mass five-body problem

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ABSTRACT

We apply our variational method to search for new periodic orbits in the planar equal-mass five-body problem. Three types of configurations are considered: all masses on a line, three masses on a line with the other two masses symmetrically locating on its two sides, and one mass on a line with the other four masses symmetrically sitting on two sides of the line. By setting suitable free variables in each pair of the three configurations and minimizing the Lagrangian action over these free variables, many new orbits were found. These orbits can be classified into 5 categories.

1. Introduction

Variational method is one of the important tools in the study of the N-body problem. It has been applied to construct periodic solutions, under various types of symmetry constraints or topological constraints [2–10,12,23]. In 2000, Chenciner and Montgomery [2] successfully showed the existence of the figure-eight orbit. A significant breakthrough in their work is to consider the action minimizer in some well-chosen symmetric loop space so that they can eliminate possible collisions in the corresponding action minimizer. Inspired by [2], many periodic solutions have been discovered and shown to exist in the N-body problem as local Lagrangian action minimizers [4–9,12,23]. The trajectories of these periodic orbits are usually symmetric. More importantly, these orbits have variational properties which can be applied to understand their linear stabilities [13,17,18]. It will be interesting if one can classify such periodic orbits into finite categories. Actually, Broucke [1] has successfully applied variational method to classify periodic orbits in the planar equal-mass four-body problem. By analyzing the symmetries of the starting configurations, he found several new sets of periodic orbits, such as a stable star-shaped choreography [12].

On the other hand, Hilbert's direct method in the calculus of variations has been well developed. Instead of concentrating on the whole part of a periodic orbit and imposing symmetry constraint, it only studies a special piece, which is an action minimizer connecting two specific configurations. For example, the action minimizer between an Euler configuration and an isosceles configuration can generate the famous figure-eight orbit. In other words, this piece of orbit contains all the symmetry information and variational properties of the figure-eight orbit. In 2002, both Marchal [10] and Chenciner [3] have

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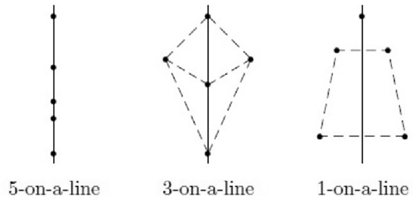

Fig. 1. 3 special configurations in the planar equal-mass five-body problem.

successfully proved that there is no collision in an action minimizer connecting two fixed ends except on the boundaries. After their celebrated works ([3,10]), the exclusion of collisions in the action minimizer can be reduced to the exclusion of boundary collisions. Many specialists have contributed to this method and apply it to prove the existence of periodic or quasi-periodic orbits. Related works can be found in [7–9,12] and references therein.

Motivated by [1,15,16] and [19], we intend to find new periodic orbits and classify them in the planar equal-mass five-body problem. The variational method we use here is based on Hilbert's direct method in the calculus of variations. In [1], Broucke searched one special type of periodic orbits in the planar five-body problems, in which one of the five bodies stays at the origin forever. In this work, we will concentrate on periodic orbits other than this special type. Instead of following the variational method in [1], we introduce a different variational approach, which can be referred to as a two-point free boundary value problem with prescribed boundary configurations [12,19]. An action minimizer can be found by minimizing the Lagrangian action of paths connecting a pair of special configurations. If one chooses appropriate boundary configurations, such an action minimizer can be one part of a periodic orbit.

By analyzing the possible symmetri shapes for the planar equal-mass five-body problem, we concentrate on three configurations as in Fig. 1: all five masses on a line (5-on-a-line in short), three masses on a line with the other two masses symmetrically locating on the two sides (3-on-a-line in short), and one mass on a line with the other four masses symmetrically locating on two sides of the line (1-on-a-line in short). By searching the six pairs of configurations, we can find many new periodic orbits. Actually, these orbits can be classified into 5 categories. It is worthy to note that the orbits found by our variational method are local action minimizers connecting two specific configurations, and some of them can be shown to exist mathematically.

The paper is organized as follows. In Section 2, we briefly introduce our variational method and compare it with the Fourier series method. Section 3 defines the three special configurations and searches for action minimizers connecting every pair of specific configurations. They are classified into 5 categories and the pictures of the corresponding periodic orbits are presented. Section 4 explains mathematically why the action minimizers exist and how they extend to periodic orbits. Conclusion and remarks are given in Section 5.

2. Variational method

Inspired by Broucke's work [1], we intend to classify action minimizing periodic orbits in the planar equal-mass five-body problem. Instead of considering only the starting configurations and searching for coefficients of Fourier series [1,14], we concentrate on a pair of symmetric configurations as the starting and ending positions respectively. Local action minimizers between two symmetric configurations could be periodic. Before presenting our result, we introduce our variational method first (see [12,21,22]).

Our method is a two-step minimizing procedure. The first step is to consider a fixed-end boundary value problem. Let $Q_s = (a_{ij}) \in \mathbb{R}_{N\times d}$ and $Q_e = (b_{ij}) \in \mathbb{R}_{N\times d}$ be two given matrices of $N \times d$, which represent two configurations of the N bodies in dimension d ($d = 1, 2, 3$). Let $\mathcal{P}(Q_s, Q_e)$ be the set of paths connecting the two given configurations in the functional space $H^1([0, 1], \chi)$, where $\chi = \left\{q(t) \in \mathbb{R}_{N\times d} \,\middle|\, \sum_{i=1}^N m_i q_i(t) = 0\right\}$ and

$$\mathcal{P}(Q_s, Q_e) = \{\mathbf{q} \in H^1([0, 1], \chi) \mid \mathbf{q}(0) = Q_s,\ \mathbf{q}(1) = Q_e\}.$$

For a fixed boundary value problem, it is known that there is a corresponding minimizer of the Lagrangian action functional $\mathcal{A}(\mathbf{q}) = \int_0^T L(\mathbf{q}, \dot{\mathbf{q}})dt$ over the path space $\mathcal{P}(Q_s, Q_e)$. By [10] and [3], this minimizer is collision free for $t \in (0, 1)$. Let

$$\mathcal{A}(Q_s, Q_e) = \inf_{\mathbf{q}\in\mathcal{P}(Q_s,Q_e)} \mathcal{A}(\mathbf{q}) = \inf_{\mathbf{q}\in\mathcal{P}(Q_s,Q_e)} \int_0^1 L(\mathbf{q}, \dot{\mathbf{q}})dt.$$

If one wants this minimizer to be a part of a periodic orbit, the given position matricies Q_s and Q_e must be quite special. For this purpose, the second step is to introduce some free variables to the two matrices Q_s and Q_e and do a second minimizing process with respect to these free variables in Q_s and Q_e. However, if Q_s and Q_e have too many free variables, there may not exist an action minimizer to reach the infimum value of the action. On the other hand, if the number of free variables on Q_s and Q_e is not enough, then the corresponding minimizer may not be periodic. We find that when the two free boundaries Q_s and Q_e satisfy a set of relation $\{(Q_s, Q_e) \mid G(Q_s, Q_e) = 0\}$, the local minimizer

$$\operatorname*{localmin}_{\{G(Q_s,Q_e)=0\}} \mathcal{A}(Q_s, Q_e) = \operatorname*{localmin}_{\{G(Q_s,Q_e)=0\}} \inf_{\mathbf{q}\in\mathcal{P}(Q_s,Q_e)} \mathcal{A}(\mathbf{q})$$