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Research paper

Noise induced transitions and topological study of a periodically driven system

Zhen Chen, Xianbin Liu*

State Key Lab of Mechanics and Control for Mechanical Structures, College of Aerospace Engineering, Nanjing University of Aeronautics and Astronautics, 29 YuDao Street, Nanjing 210016, PR China

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ABSTRACT

Noise induced transitions of an overdamped periodically driven oscillator are investigated theoretically and numerically in the limit of weak noise due to the Freidlin-Wentzell large deviation theory. Heteroclinic trajectories are found to approach the unstable orbit with fluctuational force tending to zeros. The global minimizer of the action functional corresponds to the most probable escape path and it shows a good agreement with statistical results. We then study the origins of singularities from a topological point of view by considering structures of the Lagrangian manifold and action surface. The switching line and cusp point turn out to have physical significance since they may impact the prehistory distributions, making the optimal path invalid.

1. Introduction

Dynamical systems are often subjected to random perturbations having small amplitudes. However, they may have a profound impact on the dynamics if observations are performed on a sufficient large time scale. For instance, random perturbations result in transitions between various regions around stable states of the deterministic dynamical system, which relates to the so-called metastability observed in a great number of scientific phenomenons, such as chemical reactions, climate regimes, and neuroscience. For small random perturbations, the Freidlin-Wentzel theory[1] of large deviations provides a proper framework to depict their effects on dynamics. Put simply, the theory builds on the fact that almost unlikely events, when they occur, do so with an overwhelming probability in the way that is least improbable. This makes the mechanism of these transitions predictable by using certain action functional which is the central object in the theory. Its minimum serves as the exponential rate of a stationary probability density in the approximated WKB form in the weak noise limit[2] and estimates the transition rates between stable states of the original deterministic system. Moreover, the minimizer of the action functional gives the pathway that is most probable by which the event occurs.

In past decades a great deal of work has been devoted to problems of noise induced transitions or escapes. Smelyanskiy, Dykman and Maier[3] investigated escapes from an unstable focus of a periodically oscillating system. Luchinsky[4] studied the noise driven exit in a double-well system lacking detailed balance and found a bifurcation of MPEPs as parameters varied through analog and digital stochastic simulation. In addition, noise induced escapes from chaotic attractors have also been engaging interest of researchers, such as nonhyperbolic attractors[5,6] and Lorenz attractor[7,8]. Singularities can be observed in the patterns of fluctuational paths, such as cusps and caustics[9–12] since several paths may arrive at the same

* Corresponding author.
E-mail addresses: czkillua@icloud.com (Z. Chen), xbliu@nuaa.edu.cn (X. Liu).

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terminal point. However, limited work has been done in the study of topology of Lagrangian manifold and its singularities. Smelyanskiy, Dykman and Maier[3] found the Lagrangian manifold has a novel structure with folds spiraling into the focus. A twisting structure of Lagrangian manifold was demonstrated by studying its separate cross sections along the MPEP and it was found that a birth of a cusp point gave rise to a rotation of the Lagrangian manifold[13]. Dykman[14] studied the fluctuations in a periodically driven overdamped oscillator and briefly discussed the topological properties of the patterns.

In this paper, we continue predecessors' work and investigate the problem from a topological point of view. In contrary to the previous work[11] by the authors, which only considered the fluctuations within the domain of attraction and illustrated some singular features of the pattern, we turn to investigate the noise induced transitions between the metastable states. This phase space is suspended to become $\mathbf{S}^1 \times \mathbf{R}^1$, i.e., the suspension flow is defined on a two dimensional manifold, rather than a plane. Moreover, we try to understand and explain the singular features which are not contained in [11] from a topological point of view, such the cusp point and switching line. The paper is organized as follows. In Section 2, the problem is formulated and the corresponding deterministic system is discussed. In Section 3 the WKB approximation is used. Noise induced escape is then discussed in detail by the method of action plot and several types of escape paths are given. In Section 5 the effect of cusp point and switching line on the escape is considered through the Lagrangian manifold and action surface. Conclusions are drawn in Section 6.

2. Formulation

We investigate an overdamped system driven by a periodic force $K(q;\, t)$ and white noise $\xi(t)$, the equation of which is

$$\begin{gathered} \dot{q} = K(q;t) + \xi(t), \quad K(q;t) = K(q;t+T) \\ \left\langle \xi(t)\xi\left(t'\right)\right\rangle = D\delta\left(t-t'\right) \end{gathered} \tag{1}$$

The model (1) has a wide application in many physical systems and a great deal of effort in its study has been devoted[15–17]. We consider a simple example of model (1) with

$$\dot{q} = -U'(q) + A\cos(\omega t) + \xi(t) \tag{2}$$

$$U(q) = -\frac{1}{2}q^2 + \frac{1}{4}q^4 \tag{3}$$

where A, ω are parameters. The detailed balance condition does not hold due to the presence of external harmonic driving force, with the amplitude A chosen large enough to be beyond the perturbative regime. Only the noise intensity D will be assumed small.

The presence of the external force makes the equilibrium points of the potential U into periodic states, i.e., its two minima become stable periodic cycles and the maximum becomes an unstable cycle. The one-dimensional system (2) is nonautonomous. In order to make it an autonomous system, we have the following

$$\begin{aligned} &\dot{q}_1 = \omega \\ &\dot{q}_2 = -U'(q) + A\cos(q_1) + \xi(t) \end{aligned} \tag{4}$$

The phase space of system (4) is a cylinder defined by $\mathbf{S}^1 \times \mathbf{R}^1$. Fig. 1 shows its vector field for $q_1 \in [0, 2\pi)$, with each arrow denoting the direction of the flow at corresponding point. Three nullclines are also plotted to indicate the periodic orbits, with the middle one unstable and the other two stable. We have set $A = 0.264$, $\omega = 1.2$ throughout this paper.

3. The WKB approximation

The presence of noise in system (4) may induce transitions from one stable cycle to the other one. If the noise intensity D is small, the escape from the domain of attraction follows a unique optimal trajectory with overwhelming probability, seemingly in an almost deterministic way. To calculate its probability leads us to investigate the asymptotic solution of the corresponding Fokker–Planck equation as $D \to 0$. In the limit of weak noise intensity D one can seek an approximate solution in an eikonal or WKB form

$$P(\mathbf{x}) \sim C(\mathbf{x})exp[-S(\mathbf{x})/D] \tag{5}$$

with $C(\mathbf{q})$ a prefactor not investigated in this paper and $S(\mathbf{q})$ the activation energy of fluctuations to the vicinity of the point $\mathbf{q} = (q_1, q_2)$ in the state space. $S(\mathbf{q})$ is also called quasipotential or nonequilibrium potential[1].

Substituting Eq. (5) into the Fokker-Planck equation and keeping only the terms of lowest order in D, we acquire the Hamilton–Jacobi equation for $S(\mathbf{q})$:

$$H(\mathbf{q}, \mathbf{p}) \equiv \mathbf{u}(\mathbf{x}) \cdot \mathbf{p} + \frac{1}{2}\mathbf{p}^T\mathbf{p} = 0, \mathbf{p} \equiv \frac{\partial S}{\partial \mathbf{x}} \tag{6}$$

To solve Eq. (6) one can employ the method of characteristics, arriving at the following equations:

$$\frac{d\mathbf{q}}{dt} = \frac{\partial H}{\partial \mathbf{p}} = \mathbf{u}(\mathbf{q}) + \mathbf{p} \tag{7}$$