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Research paper

Method of calculating densities for isotropic ballistic Lévy walks

Marcin Magdziarz*, Tomasz Zorawik

Hugo Steinhaus Center, Faculty of Pure and Applied Mathematics, Wrocław University of Science and Technology, Wyb. Wyspiańskiego 27, 50-370 Wrocław, Poland

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ABSTRACT

We provide explicit formulas for asymptotic densities of d-dimensional ($d > 1$) isotropic Lévy walks in a ballistic regime. The densities of multidimensional undershooting and overshooting Lévy walks are presented as well. Interestingly, when the number of dimensions is odd the densities of all these Lévy walks are given by elementary functions. When d is even, we can express the densities as fractional derivatives of hypergeometric functions, which makes an efficient numerical evaluation possible. A simulation algorithm for isotropic Lévy walks is presented as well. The theoretical results are in agreement with the results of Monte Carlo simulations.

1. Introduction

Lévy walks are one of the most important tools for modeling anomalous stochastic transport phenomena. The period of intensive research on this topic started with the pioneering papers [1] by Shlesinger et al. and [2] by Klafter et al. Since then Lévy walks found many applications in different areas of physics and biology. The list of real-life phenomena and complex systems where Lévy walks are used includes (but is not limited to) migration of swarming bacteria [3], blinking nanocrystals [4], light transport in optical materials [5], fluid flow in a rotating annulus [6], foraging patterns of animals [7–9], human travel [10,11] and epidemic spreading [12,13]. For more background information about Lévy walks and their applications the interested reader is referred to the recent review [14].

Lévy walks exhibit two important features. The first one is a power-law jump distribution and the second one is that a distribution of a particle's position has finite moments of all orders. This is obtained by introducing a dependence between waiting times and lengths of the jumps – we require that the length of the jump be equal to the preceding waiting time in the underlying continuous-time random walk (CTRW) scenario. The continuity of the trajectories is obtained by a linear interpolation of the corresponding CTRW. As a result a velocity v of a particle performing Lévy walk is constant. The appearance of this two features together – the power-law jump distribution and the finiteness of all moments of the distribution of the particle's position – stays in contrast with a different very popular model for anomalous transport, namely Lévy flights [15–17]. For Lévy flights the power-law jump distribution implies that the mean square displacement is infinite. For other correlated fractional diffusion models we refer the reader to [18–26]

In spite of the long history and popularity of Lévy walks, their multidimensional probability density functions (PDFs) were not known. In this paper we fill this gap and derive explicit PDFs of d-dimensional Lévy walks, where $d > 1$. When d

* Corresponding author.
E-mail addresses: Marcin.Magdziarz@pwr.wroc.pl (M. Magdziarz), tomasz.zorawik@pwr.edu.pl (T. Zorawik).

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is an odd number $d = 2n + 3$ ($n \in \mathbb{N}$), the PDF is expressed by elementary functions. This fact is somehow unexpected since the limit process for Lévy walks is given as a composition of certain α-stable processes, and it is a known fact that the stable densities are expressed by elementary functions only for special values of α [17]. Here the PDFs of Lévy walks will be expressed in terms of elementary functions for all $\alpha \in (0, 1)$. In the case when the dimension is even $d = 2n + 2$ ($n \in \mathbb{N}$) the PDF can also be computed, but the formula involves hypergeometric functions and the Riemann–Liouville right-side fractional derivative, which can be efficiently evaluated numerically. We also provide explicit formulas for the densities of other coupled CTRWs – the so-called undershooting and overshooting Lévy walks (also known as wait-first and jump-first Lévy walks [14]). Similarly, these densities are given by elementary functions for odd dimensions d. Moreover these PDFs solve certain fractional differential equations [27,39] with a fractional material derivative [30,31]. In [39] an algorithm of approximating those equations via Monte Carlo methods was proposed. Here we derive the analytical solutions and compare them in Section 5 with the numerical estimations. We also present an algorithm to numerically generate continuous Lévy walks based on [39].

The densities of 1-dimensional ballistic Lévy walks have been found by Froemberg et al. in [32], see also [33] for other approach to this problem and [4,28,29] for the closely related issue of finding the distribution of time averages for processes. Recently, in [34] the PDFs of 2 and 3-dimensional ballistic Lévy walks were derived. Comparison of three different models of Lévy walks in two dimensions can be found in [35]. In this paper we calculate the densities of d-dimensional Lévy walks when $d > 1$ is arbitrary. We also apply our method to the overshooting and undershooting Lévy walks. The main idea behind this method is to take advantage of the rotational invariance of the Lévy walks and connect the multi-dimensional PDF with a proper one-dimensional distribution using methods from [36], then apply the formula of Godrèche and Luck from [37] to invert the Fourier–Laplace transform.

2. Lévy walks and their limits

In this section we recall the definition of the d-dimensional standard, undershooting and overshooting Lévy walks (see [14]). We also recall their limit processes. The differences between those processes are also discussed.

2.1. Definition of Lévy walks

Let T_i ($i \in \mathbb{N}$) be a sequence of waiting times. We assume that T_i are independent, identically distributed (IID) positive random variables with power-law distribution $P(T_i > t) \propto At^{-1-\alpha}$ with $\alpha \in (0, 1)$ and $A > 0$. It follows that $n^{-1/\alpha}\sum_{i=1}^{n} T_i$ converges in distribution to T when $n \to \infty$, where T is an α-stable random variable supported on $[0, \infty)$ with Laplace transform $\mathbf{E}(\exp\{-sT\}) = \exp\{-\Gamma[1-\alpha]As^{\alpha}\}$. We set the scale parameter to $A = \frac{1}{\Gamma(1-\alpha)}$. Denote by $N(t) = \max\{k \geq 0 : \sum_{i=1}^{k} T_i \leq t\}$ the corresponding process counting the number of jumps up to time t. Next, let us define a sequence of consecutive jumps

$$\mathbf{X}_i = vT_i\mathbf{V}_i, \quad i = 1, 2, \ldots .$$

Here, $\{\mathbf{V}_i\}$ is a sequence of IID random unit vectors distributed uniformly on the $d-1$ - dimensional hypersphere $\mathbb{S}^{d-1}$. Each vector $\mathbf{V}_i$ governs the direction of ith jump. The constant v is the velocity, for simplicity it is assumed that $v = 1$. In the Introduction we mentioned that for Lévy walk the length of each jump should be equal to the corresponding waiting time. It is clear from the above equation that this condition is satisfied: $\|\mathbf{X}_i\| = T_i$, where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$. Now, the undershooting Lévy walk (or wait-first Lévy walk) is defined as

$$\mathbf{L}_{ULW}(t) = \sum_{i=1}^{N_t} \mathbf{X}_i. \tag{1}$$

This is a CTRW, so the trajectories are piecewise constant and have jumps. The overshooting Lévy walk (or jump-first Lévy walk) has the following definition

$$\mathbf{L}_{OLW}(t) = \sum_{i=1}^{N_t+1} \mathbf{X}_i. \tag{2}$$

This is also a CTRW so the trajectories of this process are also not continuous. However, applying simple linear interpolation on the trajectories of $\mathbf{L}_{ULW}(t)$ and $\mathbf{L}_{OLW}(t)$, we arrive at the final definition of the standard Lévy walk $\mathbf{L}(t)$:

$$\begin{aligned}\mathbf{L}(t) &= \mathbf{L}_{ULW} + \left(t - \sum_{i=1}^{N(t)} T_i\right)(\mathbf{L}_{OLW} - \mathbf{L}_{ULW}) \\ &= \sum_{i=1}^{N(t)} T_i\mathbf{V}_i + \left(t - \sum_{i=1}^{N(t)} T_i\right)\mathbf{V}_{N(t)+1},\end{aligned} \tag{3}$$

The trajectories of $\mathbf{L}(t)$ are continuous and piecewise linear, which means that the walker moves with constant velocity v. The spatio-temporal coupling ensures that all moments, in particular a mean square displacement, are finite. This follows from that fact that $\|\mathbf{L}(t)\| \leq t$. Similarly, all the moments of $\mathbf{L}_{ULW}(t)$ are also finite. However the overshooting Lévy walk does