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Research paper

Lie symmetry analysis of the Heisenberg equation

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ABSTRACT

The Lie symmetry analysis is performed on the Heisenberg equation from the statistical physics. Its Lie point symmetries and optimal system of one-dimensional subalgebras are determined. The similarity reductions and invariant solutions are obtained. Using the multipliers, some conservation laws are obtained. We prove that this equation is nonlinearly self-adjoint. The conservation laws associated with symmetries of this equation are constructed by means of Ibragimov's method.

1. Introduction

Lie symmetry analysis method has recently penetrated into most of the areas related to mathematics, such as differential equations [1], Lie algebras [2,3], classical mechanics [4] and rogue wave [5]. As a matter of fact, many practical problems arising from natural phenomenon and engineering technology can be modeled by the nonlinear partial differential equation (NLPDEs). Investigating the explicit solutions of NLPDEs is beneficial for solving these problems. Lie symmetry analysis method can be regarded as one of the most effective methods to derive the explicit solutions. In addition, on the base of symmetries, the integrability of the NLPDEs, such as group classification, optimal system and conservation laws, can be considered successively [6–8].

The conservation law has drawn great attentions of the mathematical physicists. In the past decades, many methods for dealing with the conservation laws are derived, such as the multiplier approach [9], Noether's approach, the partial Noether's approach [10], Ibragimov's method [11]. Noether's approach and the partial Noether's approach produce a connection between the symmetries of NLPDEs and the conservation laws. However, they are not applicable to the nonlinear partial differential equations that do not admit a Lagrangian. In order to overcome such difficulties, Ibragimov's method was proposed [11]. Ibragimov's method does not require the existence of a Lagrangian and it is based on the concept of an adjoint equation for the nonlinear equations. The resultant conservation laws involve not only the solutions of the original equation, but also the solutions of the adjoint equation. If the equation under consideration is nonlinear self-adjoint [12,13], the conservation vectors obtained are the conservation vectors of the original equation.

In this paper, we consider the Heisenberg equation [14] of the form

$$\begin{aligned}(u_t - u_{xx})(u+v) + 2u_x^2 &= 0,\\ (v_t + v_{xx})(u+v) - 2v_x^2 &= 0,\end{aligned} \tag{1}$$

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where the subscripts denote the partial derivatives, $u(x, t)$ and $v(x, t)$ are functions with two independent variables x, t and $u(x,t)+v(x,t)\neq 0$. This equation relates to the famous Heisenberg model $S_t=[S,S_{xx}]$ via the stereographic projection [14]. Heisenberg equation is one of the models used in statistical physics to model ferromagnetism and other phenomena.

The present paper aims at obtaining the optimal system, similarity reductions, invariant solutions and conservation laws of the Heisenberg equation. The organization of the paper is as follows. In Section 2, we obtain the Lie point symmetries of the Heisenberg equation using Lie group analysis. Section 3 devotes to constructing an optimal system of one-dimensional subalgebra. In Section 4, we consider the similar reductions and group-invariant solutions of this equation. In Section 5, three local conservations laws of Eq. (1) are obtained via the multipliers. In Section 6, Eq. (1) is proved nonlinearly self-adjoint. The conservation laws of Eq. (1) are established in Section 7 using Ibragimov's method. The last section contains a summary and discussion.

2. Lie point symmetries

In this section, we perform Lie symmetry analysis on the Heisenberg equation. Consider a one-parameter Lie group of transformations

$$\begin{aligned}
&t\to t+\varepsilon\xi^1(x,t,u,v),\\
&x\to x+\varepsilon\xi^2(x,t,u,v),\\
&u\to u+\varepsilon\eta^1(x,t,u,v),\\
&v\to v+\varepsilon\eta^2(x,t,u,v),
\end{aligned}$$

with a small parameter $\varepsilon\ll 1$. The corresponding generator of the Lie algebra is of the form

$$X=\xi^1(t,x,u,v)\frac{\partial}{\partial t}+\xi^2(t,x,u,v)\frac{\partial}{\partial x}+\eta^1(t,x,u,v)\frac{\partial}{\partial u}+\eta^2(t,x,u,v)\frac{\partial}{\partial v}.$$

Thus the second prolongation $pr^{(2)}X$ is

$$pr^{(2)}X=X+\eta^1_t\frac{\partial}{\partial u_t}+\eta^2_t\frac{\partial}{\partial v_t}+\eta^1_x\frac{\partial}{\partial u_x}+\eta^2_x\frac{\partial}{\partial v_x}+\eta^1_{xx}\frac{\partial}{\partial u_{xx}}+\eta^2_{xx}\frac{\partial}{\partial v_{xx}},$$

where

$$\begin{aligned}
&\eta^1_t=D_t(\eta^1)-u_tD_t(\xi^1)-u_xD_t(\xi^2),\\
&\eta^2_t=D_t(\eta^2)-v_tD_t(\xi^1)-v_xD_t(\xi^2),\\
&\eta^1_x=D_x(\eta^1)-u_tD_x(\xi^1)-u_xD_x(\xi^2),\\
&\eta^2_x=D_x(\eta^2)-v_tD_x(\xi^1)-v_xD_x(\xi^2),\\
&\eta^1_{xx}=D_x(\eta^1_x)-u_{xt}D_x(\xi^1)-u_{xx}D_x(\xi^2),\\
&\eta^2_{xx}=D_x(\eta^2_x)-v_{xt}D_x(\xi^1)-v_{xx}D_x(\xi^2),
\end{aligned}$$

and the operators D_x and D_t are the total derivatives with respect to t and x. The determining equations of Eq. (1) arises from the following invariance condition

$$\begin{aligned}
&pr^{(2)}X(\Delta_1)|_{\Delta_1=0}=0,\\
&pr^{(2)}X(\Delta_2)|_{\Delta_2=0}=0,
\end{aligned}$$

where $\Delta_1=(u_t-u_{xx})(u+v)+2u_x^2=0,\quad \Delta_2=(v_t+v_{xx})(u+v)-2v_x^2=0$. Then we obtain the overdetermined system of the partial differential equations

$$\begin{aligned}
&\xi^1_u=0,\quad \xi^1_v=0,\quad \xi^1_x=0,\quad \xi^1_{tt}=0,\\
&\xi^2_t=0,\quad \xi^2_u=0,\quad \xi^2_v=0,\quad \xi^2_x-\frac{1}{2}\xi^1_t=0,\\
&\eta^2_t=0,\quad \eta^2_u=0,\quad \eta^2_x=0,\quad \eta^2_{vvv}=0,\\
&\eta^1=-\frac{1}{2}(u+v)^2\eta^2_{vv}+(u+v)\eta^2_v-\eta^1.
\end{aligned}$$

Solving the system, one can get

$$\begin{aligned}
&\xi^1=c_1t+c_5,\\
&\xi^2=\frac{c_1}{2}x+c_6,\\
&\eta^1=-\frac{c_3}{2}u^2-c_4+c_2u,\\
&\eta^2=\frac{c_3}{2}v^2+c_2v+c_4,
\end{aligned}$$