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Research paper

Conservative modified Serre–Green–Naghdi equations with improved dispersion characteristics

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ABSTRACT

For surface gravity waves propagating in shallow water, we propose a variant of the fully nonlinear Serre–Green–Naghdi equations involving a free parameter that can be chosen to improve the dispersion properties. The novelty here consists in the fact that the new model conserves the energy, contrary to other modified Serre's equations found in the literature. Numerical comparisons with the Euler equations show that the new model is substantially more accurate than the classical Serre equations, especially for long time simulations and for large amplitudes.

1. Introduction

Water waves in channels and oceans are usually described by the Euler equations. Due to their complexity, several approximate models have been derived in various wave regimes. Once a new mathematical model is proposed, the limits of its applicability have to be determined. In shallow water, the main restriction comes from the ratio between the characteristic wavelength λ and the mean water depth d, the so-called shallowness parameter $\sigma = d/\lambda \ll 1$. Restrictions on the free surface elevation are characterised by the dimensionless parameter $\epsilon = a/d$, where a is a typical amplitude ($\epsilon\sigma = a/\lambda$ is a wave steepness). Many approximate equations have been derived for waves in shallow water, such as the Korteweg–deVries (KdV) equation [22] for unidirectional waves, the Saint-Venant equations [39] for bidirectional non-dispersive waves and many variants of the Boussinesq equations [4,6] for dispersive waves propagating in both directions. In addition to shallowness ($\sigma \ll 1$), KdV and Boussinesq equations assume small amplitudes (e.g. $\epsilon = \mathcal{O}(\sigma^2)$) [19].

Considering long waves propagating in shallow water but without assuming small amplitudes (i.e. $\sigma \ll 1$ and $\epsilon = \mathcal{O}(1)$), Serre [35] derived a so-called *fully-nonlinear weakly dispersive* system of equations [40,41] which, after further approximations, include the Korteweg-deVries, Saint-Venant and Boussinesq equations as special cases. For steady flows, these equations were already known to Rayleigh [26]. The Serre equations were independently rediscovered by Su and Gardner [38], and again later by Green, Laws and Naghdi [18]. These nowadays popular equations constitute an asymptotic fully nonlinear model including all the terms up to order σ^3 into the momentum equation. Serre's equations represent a substantial

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improvement with respect to the Boussinesq theory [8], but many shallow water phenomena involve significant dispersive effects that are not well described by Serre's equations.

One possibility to improve the Serre model is by including the terms of order σ^5 (and higher-order terms) into the momentum equation. This program was accomplished for Boussinesq equations in, e.g., [28]. However, this modification makes the fifth-order (and higher-order) derivatives to appear in the model equations, making its numerical resolution (and thus its applicability) rather challenging. Actually, the numerical resolution of these high-order Boussinesq-like equations is slower (and less accurate) than that of the Euler equations, at least for simple (periodic) domains.

Another way of improving the classical model was coined by Bona and Smith [5] (see also [4,34]). The idea consists in introducing a free parameter into the model that can be appropriately chosen to improve some of the desired properties. This can be achieved by replacing the depth-averaged velocity variable by the velocity of the fluid evaluated at a certain depth in the bulk of fluid [29]. Most often, this parameter is chosen to optimise somehow the dispersion relation properties [20,27]. In general, the community is rather focused on various linear properties of the model, even if it is used later to simulate nonlinear waves.

Similar ideas have also been applied to the Serre equations with flat [12,15,25] and varying bottoms [1,2,8]. Free parameters are obtained from arbitrarily-weighted averages of different (but of same order) approximations of some quantities. However, these modified Serre equations invalidate one fundamental physical property: the conservation of the energy. Some variants also invalidate the Galilean invariance [13], that is a severe drawback for many applications. The same remarks apply as well to previous works on the improved Boussinesq equations [3,30]. Thus, one may improve the dispersive properties of the model but, on the other hand, loses the energy conservation property. For many applications, especially in the case of long time simulations, the disadvantages can be crucial, overriding all the possible advantages. In the present paper, we address this issue, proposing a method for deriving an improved version of the Serre equations that preserves the aforementioned nice properties of the original Serre model. For the sake of simplicity, the method is illustrated for 2D waves over a horizontal bottom, but generalisations to 3D and varying bottom can be obtained in an analogous manner.

The present paper is organised as follows. In Section 2 a simple derivation of the classical Serre equations is presented. It is followed by the derivation of an already known one-parameter generalisation of these equations and its shortcomings are explained. In Section 3 we derive a new one-parameter generalisation of the Serre equations that conserve the energy. In Section 4 we discuss several criteria for choosing the free parameter. Some numerical results are provided in Section 5 demonstrating the advantages of the new Serre-like equations. Finally, the main conclusions, possible generalisations and perspectives are discussed in Section 6.

2. Classical and modified Serre's equations

Here, we derive the classical Serre equations and a modified version of these equations with an additional free parameter using variational methods.

2.1. Classical Serre's equations

In order to model irrotational two-dimensional long waves propagating in shallow water over horizontal bottom, one can approximate the velocity field by the ansatz

$$u(x,y,t) \;\approx\; \bar{u}(x,t), \qquad v(x,y,t) \;\approx\; -(y+d)\,\bar{u}_x, \tag{1}$$

where d is the water depth and $\bar{u}$ is the horizontal velocity averaged over the water column — i.e., $\bar{u} = h^{-1}\int_{-d}^{\eta} u\,\mathrm{d}y$, $h = \eta + d$ the total water depth — $y = \eta$ and $y = 0$ being the equations of the free surface and of the still water level, respectively; a sketch of the fluid domain is depicted in the Fig. 1. The horizontal velocity u is thus (approximately) uniform along the water column and the vertical velocity v is chosen so that the fluid incompressibility is valid. Note that the vorticity $\omega = v_x - u_y \approx -(y+d)\bar{u}_{xx}$ is not exactly zero, meaning that the potential flow is approximated by a rotational velocity field. The fact that the irrotationality is violated should not be more surprising than the violation of other relations, such as the isobarity of the free surface.

With the ansatz (1), the vertical acceleration is

$$\frac{\mathrm{D}\,v}{\mathrm{D}t} \;=\; \frac{\partial\,v}{\partial t} \;+\; u\,\frac{\partial\,v}{\partial x} \;+\; v\,\frac{\partial\,v}{\partial y} \;\approx\; -\,v\,\bar{u}_x \;-\; (y+d)\,\frac{\mathrm{D}\,\bar{u}_x}{\mathrm{D}t} \;=\; \gamma\,\frac{y+d}{h}, \tag{2}$$

where γ is the vertical acceleration at the free surface, i.e.,

$$\gamma \;=\; \left.\frac{\mathrm{D}\,v}{\mathrm{D}t}\right|_{y=\eta} \;\approx\; h\big[\,\bar{u}_x^2 \;-\; \bar{u}_{xt} \;-\; \bar{u}\,\bar{u}_{xx}\,\big] \;=\; 2\,h\,\bar{u}_x^2 \;-\; h\,\partial_x[\,\bar{u}_t \;+\; \bar{u}\,\bar{u}_x\,]. \tag{3}$$

The kinetic and potential energies per water column, respectively $\mathscr{K}$ and $\mathscr{V}$, are similarly easily derived

$$\frac{\mathscr{K}}{\rho} \;=\; \int_{-d}^{\eta} \frac{u^2+v^2}{2}\,\mathrm{d}y \;\approx\; \frac{h\,\bar{u}^2}{2} \;+\; \frac{h^3\,\bar{u}_x^2}{6}, \qquad \frac{\mathscr{V}}{\rho} \;=\; \int_{-d}^{\eta} g\,(y+d)\,\mathrm{d}y \;=\; \frac{g\,h^2}{2}, \tag{4}$$