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Well-balanced methods for the shallow water equations in spherical coordinates☆

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The goal of this work is to obtain a family of explicit high order well-balanced methods for the shallow water equations in spherical coordinates. Application of shallow water models to large scale problems requires the use of spherical coordinates: this is the case, for instance, of the simulation of the propagation of a Tsunami wave through the ocean. Although the PDE system is similar to the shallow water equations in Cartesian coordinates, new source terms appear. As a consequence, the derivation of high order numerical methods that preserve water at rest solutions is not as straightforward as in that case. Finite Volume methods are considered based on a first order path-conservative scheme and high order reconstruction operators. Numerical methods based on these ingredients have been successfully applied previously to the nonlinear SWEs in Cartesian coordinates. Some numerical tests to check the well-balancing and high order properties of the scheme, as well as its ability to simulate planetary waves or tsunami waves over realistic bathymetries are presented.

1. Introduction

The shallow water equations (SWEs) are useful to model free surface gravity waves whose wavelength is much larger than the characteristic bottom depth: see [39–41] for a review of these equations. This is the case of tsunami waves: although the depth bottom of the oceans cannot be considered as small, the characteristic wavelength of a tsunami can be of the order of 100 km, what is significantly larger than the characteristic ocean depth.

On the other hand, the application of SWE to large scale phenomena (of the order of 1000s of km) makes necessary to take into account the curvature of the Earth. Usually, the Earth is approached by a sphere and the equations are written in spherical coordinates. Although the PDE system is similar to the SWEs in the plane using Cartesian coordinates, new source terms appear due to the change of variables. Therefore, the discretization of the system in spherical coordinates goes far beyond a simple adaptation of the numerical methods for the equations written in Cartesian coordinates.

SWEs in spherical coordinates are the basis of many of the most used software packages for tsunami simulations. In most cases, the linear SWEs are considered, what is enough to give an acceptable simulation of the propagation of the wave in deep waters: [2,32]. Nevertheless, the linear SWEs cannot be used for the simulation of the arrival of the wave to the shore and the subsequent flooding. On the other hand, when the nonlinear SWEs are considered, in most cases the formulation in primitive variables (i.e. velocity/thickness) is used instead of the conserved ones (discharge/thickness): see [31]. While the systems written in one or another set of variables is equivalent for smooth solutions, this is not the case when shock waves develop: the jump condition depends on the formulation, and the one consistent with the physics of the system is the one corresponding to the formulation in conserved variables. Again, while the formation of shock waves is not expected during the propagation of the wave, it is very likely to happen when the wave is close to the shore.

Finally, in some cases the nonlinear SWEs in spherical coordinates using the conserved variables is used, but some of the source terms due to the change of variables are neglected, as their influence is not relevant far enough of the poles (see [32]).

In this article, we consider the nonlinear SWEs in conserved variables formulation with all the source terms related to bottom variations and to the curvature. Neither Coriolis force (whose influence is not relevant for Tsunami waves) nor friction forces (whose numerical treatment can be done like in the Cartesian coordinates case) are considered.

Our goal is to derive an explicit high order well-balanced numerical scheme. By well-balanced, we mean that stationary solutions corresponding to water at rest situations have to be pre-

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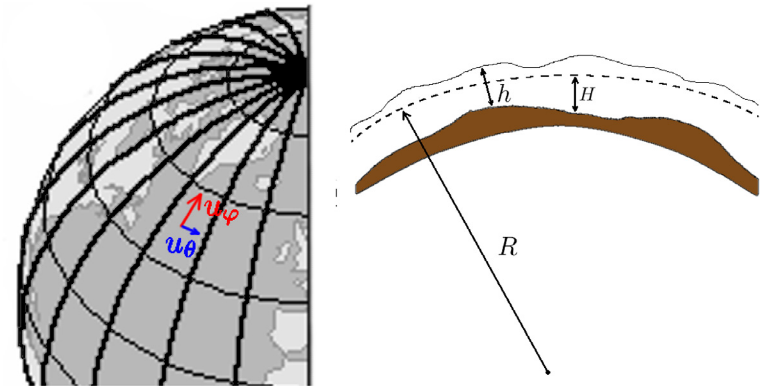

Fig. 1. Sketch of the unknowns for the shallow-water equations in spherical coordinates.

served by the numerical methods, what is a standard requirement in the context of SWEs: [1,3,4,12,22,23,26,27,30,33,35,42], among others. Finite Volume methods considered here are based on a first order path-conservative scheme for the standard 1d SWEs and on high order reconstruction operators. Numerical methods based on these ingredients have been successfully applied previously to the nonlinear SWEs in Cartesian coordinates: [6,7,15,17,18,27]. A recent review on these methods can be found in [8].

The organization of the article is as follows: in next section, the PDE system is introduced and an equivalent formulation is obtained which is better suited to the application of Finite Volume methods. In Section 3 the general form of the semidiscrete in space numerical method is introduced. Next, a general result concerning the well-balanced property of the method is proved: if the first order path-conservative scheme and the reconstruction operator are both well-balanced (in a sense to be specified) then the high order numerical method is well-balanced as well. A general way of obtaining well-balanced first order schemes and well-balanced high order reconstruction operators based respectively on [11] and [9] is also introduced. We also discuss the influence of the quadrature formulas used to approximate the volume and line integrals of the scheme in the well-balancing property. Some numerical tests are introduced in Section 5 in order to check the well-balancing and high order properties of the method, as well as its ability to simulate planetary waves or tsunami waves over a realistic bathymetry of the Mediterranean Sea. Finally, some conclusions are drawn.

2. PDE System

The shallow water equations on the sphere writes in spherical coordinates as follows (see [40] and references therein)

$$\begin{cases} \partial_t h + \dfrac{1}{R\cos(\varphi)}\Big(\partial_\theta (hu_\theta) + \partial_\varphi (hu_\varphi \cos(\varphi))\Big) = 0, \\ \partial_t u_\theta + \dfrac{u_\theta}{R\cos(\varphi)}\partial_\theta u_\theta + \dfrac{u_\varphi}{R}\partial_\varphi u_\theta - \dfrac{u_\theta u_\varphi}{R}\tan(\varphi) + \dfrac{g}{R\cos(\varphi)}\partial_\theta h \\ \qquad = \dfrac{g}{R\cos(\varphi)}\partial_\theta H, \\ \partial_t u_\varphi + \dfrac{u_\theta}{R\cos(\varphi)}\partial_\theta u_\varphi + \dfrac{u_\varphi}{R}\partial_\varphi u_\varphi + \dfrac{u_\theta^2}{R}\tan(\varphi) + \dfrac{g}{R}\partial_\varphi h = \dfrac{g}{R}\partial_\varphi H, \end{cases} \tag{1}$$

where R is the radius; (θ, φ), the longitude and latitude; g, the gravity; h, the thickness of the water layer; H, the bottom depth; u_θ, u_φ, are the longitudinal and latitudinal velocities averaged in the normal direction (Fig. 1).

As it is well known, when shock waves develop, the Rankine–Hugoniot conditions depend on the formulation of the system. Moreover, the Rankine–Hugoniot conditions related to the system written in velocity-thickness formulation are not the physically correct jump conditions. Therefore, we will consider the discharge-thickness formulation

$$\begin{cases} \partial_t h + \dfrac{1}{R\cos(\varphi)}\Big(\partial_\theta q_\theta + \partial_\varphi (q_\varphi \cos(\varphi))\Big) = 0, \\ \partial_t q_\theta + \dfrac{1}{R\cos(\varphi)}\partial_\theta\left(\dfrac{q_\theta^2}{h}\right) + \dfrac{1}{R}\partial_\varphi\left(\dfrac{q_\theta q_\varphi}{h}\right) - 2\dfrac{q_\theta q_\varphi}{Rh}\tan(\varphi) \\ \qquad + \dfrac{gh}{R\cos(\varphi)}\partial_\theta h = \dfrac{gh}{R\cos(\varphi)}\partial_\theta H, \\ \partial_t q_\varphi + \dfrac{1}{R\cos(\varphi)}\partial_\theta\left(\dfrac{q_\varphi q_\theta}{h}\right) + \dfrac{1}{R}\partial_\varphi\left(\dfrac{q_\varphi^2}{h}\right) \\ \qquad + \dfrac{(q_\theta^2 - q_\varphi^2)}{hR}\tan(\varphi) + \dfrac{gh}{R}\partial_\varphi h = \dfrac{gh}{R}\partial_\varphi H, \end{cases} \tag{2}$$

where

$$q_\theta = hu_\theta, \quad q_\varphi = hu_\varphi.$$

In order to write the equations in the form of a system of balance laws, the following variables are introduced

$$h_\sigma = h\cos(\varphi), \quad H_\sigma = H\cos(\varphi), \quad \eta_\sigma = h_\sigma - H_\sigma,$$
$$Q_\varphi = \cos(\varphi)q_\varphi, \quad Q_\theta = \cos(\varphi)q_\theta,$$

and the system is rewritten as follows:

$$\begin{cases} \partial_t h_\sigma + \dfrac{1}{R}\left(\partial_\theta\left(\dfrac{Q_\theta}{\cos(\varphi)}\right) + \partial_\varphi Q_\varphi\right) = 0, \\ \partial_t Q_\theta + \dfrac{1}{R}\partial_\theta\left(\dfrac{Q_\theta^2}{h_\sigma\cos(\varphi)}\right) + \dfrac{1}{R}\partial_\varphi\left(\dfrac{Q_\theta Q_\varphi}{h_\sigma}\right) \\ \qquad - \dfrac{Q_\theta Q_\varphi}{Rh_\sigma}\tan(\varphi) + \dfrac{gh_\sigma}{R\cos^2(\varphi)}\partial_\theta \eta_\sigma = 0, \\ \partial_t Q_\varphi + \dfrac{1}{R}\partial_\theta\left(\dfrac{Q_\varphi Q_\theta}{h_\sigma\cos(\varphi)}\right) + \dfrac{1}{R}\partial_\varphi\left(\dfrac{Q_\varphi^2}{h_\sigma}\right) \\ \qquad + \left(\dfrac{Q_\theta^2}{Rh_\sigma} + \dfrac{gh_\sigma\eta_\sigma}{R\cos(\varphi)}\right)\tan(\varphi) + \dfrac{gh_\sigma}{R\cos(\varphi)}\partial_\varphi \eta_\sigma = 0. \end{cases} \tag{3}$$

Remark 1. Since $\cos(\varphi)$ is continuous, it can be easily checked that the Rankine–Hugoniot conditions corresponding to systems (2) and (3) are equivalents. Moreover, if H is assumed to be smooth, then the products $h_\sigma\partial_\theta\eta_\sigma$ and $h_\sigma\partial_\varphi\eta_\sigma$ are well defined.

Remark 2. Let O be a domain in the $\theta - \varphi$ plane such that q_θ and q_φ vanish on its boundary. Then, the first equation implies that

$$R\int_O h_\sigma\, d\mathbf{x} = \int_O hR\cos(\varphi)\, d\theta d\varphi = \int_{\mathcal{O}} h\, d\gamma$$

is preserved, where $\mathcal{O}$ is the image of O on the sphere of radius R. Notice that this quantity is different from the water mass, whose