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# EBR-WENO scheme for solving gas dynamics problems with discontinuities on unstructured meshes

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ABSTRACT

We develop the Edge-Based Reconstruction WENO scheme for solving Euler equations on unstructured meshes. It belongs to the class of edge-based schemes with quasi-1D reconstruction of variables. The scheme monotonization is provided by using a convex combination of three lower-order reconstructions of variables in a similar way as it is in the classical finite-difference WENO scheme. The new scheme damps oscillations near shocks on unstructured meshes and, due to its edge-based nature, requires rather low computational costs. The properties of the new scheme are demonstrated on several test problems.



## 1. Introduction

When simulating a great part of gas dynamics problems, it is crucial to provide a high accuracy of the solution in smooth regions together with an adequate representation of shock waves and other discontinuities. One of the widely used methods for solving industry-oriented problems with shocks is the second-order Godunov-type MUSCL method, put forward by Kolgan [1,2], see also [3,4], and van Leer scheme [5]. In this method, numerical fluxes are determined by solving the Riemann problem with respect to the linearly reconstructed physical or conservative variables. The reconstruction slope is calculated by applying minmod-function to the corresponding upwind and downwind differences. On Cartesian meshes the Kolgan-van Leer scheme allows to avoid non-physical oscillations up- and downstream the shock front, however it also introduces too much dissipation on smooth solutions.

The Weighed essentially non-oscillatory (WENO) scheme represents a more efficient approach which allows combining higher accuracy for smooth solutions with the correct treatment of shocks. In this scheme, the left and right states involved in the Riemann problem are found with the use of linear combination of some reconstructed values. The weights of the linear combinations are estimated basing on the integral characteristics of corresponding interpolation polynomials. The WENO scheme was first designed for 1D problems in [6]. The paper [7] introduced new smoothness monitors and extended the method to the multidimensional case.

In the multidimensional case on Cartesian meshes, there exist two implementations of the WENO scheme, namely, finite-difference and finite-volume approaches. In the finite-difference WENO scheme (FD WENO) which was proposed in [7] the values of variables are interpreted as point values in nodes or cell centers whereas their derivatives along the coordinate lines are evaluated independently using the 1D-WENO scheme. This approach is cost-efficient since the complexity grows linearly with the problem dimension. The finite-volume WENO scheme (FV WENO) assumes the definition of data as integral mean values over mesh elements. Thus, to calculate the numerical flux through each face of a mesh element, one needs to build a multidimensional polynomial. As a result, the FV WENO scheme in the nonlinear case becomes significantly more expensive than the FD WENO version. A detailed description and comparison of the FD and FV WENO schemes is given in [8]. These schemes are also compared in [9] where the FD WENO method is called the FV WENO scheme of class A, and the polynomial-based FV WENO method – the WENO scheme of class B.

It should be noted that despite the noticeably higher costs of the finite-volume approach on a Cartesian mesh in comparison with the finite-difference analogue, it offers a straightforward generalization to unstructured meshes as it was carried out for smooth solutions in [10].

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The idea to use weighted combinations of polynomials on unstructured meshes similarly as in the WENO scheme was first implemented in [11] and [12], however the full-fledged WENO scheme for unstructured meshes was developed in the paper of Hu and Shu [13]. In a series of follow-up papers [14–16], in order to improve the efficiency of WENO-scheme implementation on unstructured meshes, the authors conducted additional investigations of the FV WENO method on multidimensional Cartesian meshes. The further developments of ENO and WENO schemes on unstructured meshes are represented in the papers [17–23] and some others. In spite of a great number of works on polynomial-based FV WENO schemes on unstructured meshes, their high computational costs still remain a significant shortcoming.

The edge-based schemes represent a special class of finite-volume methods on unstructured meshes. Their distinctive feature is the definition of flow variables as point values rather than cell-averages, whereas the flux values are evaluated at the edge midpoints only. The edge-based schemes have significantly lower computational costs than the multidimensional polynomial-based k-exact methods, however within this approach it is not possible to build a scheme of arbitrary high order of approximation. Among the most known edge-based schemes there are the higher-accuracy scheme of T. Barth [24], the Flux Correction (FC) method [25–29] and the Edge-Based Reconstruction (EBR) schemes [30–35] which exploit quasi-1D reconstructions of variables.

In the EBR schemes the edge-based quasi-1D reconstructions of variables are built based on the edge vertices and several additional points on the edge-based line where the values are calculated using linear interpolations on the corresponding intersecting faces. The quasi-1D approach was first proposed in [30] where, besides the edge vertices, the authors use two extra points: one from each side of the parent edge. Later on in [31] it was suggested to extend the stencils by taking not only these two extra points, but also the gradients in the vertices that are involved in the value interpolations to the additional points. The resulting schemes were LV5 and NLV5 in dependence on a type of reconstructed variables. The stencil extension improves the dissipative and dispersive properties of the method on unstructured meshes. Instead of using the gradients, the paper [35] put forward the SEBR5 scheme which involves 4 additional points: two from each side of the edge. The values in these additional points are found using linear interpolations. A reduced version of the SEBR5 scheme with 2 additional points (one from each side of the edge) is denoted below as the EBR3 scheme.

The quasi-1D reconstructions are built in the way that provides the 3rd and 5th order of approximation for EBR3 and SEBR5 schemes respectively on translationally-invariant (TI) meshes (i.e. meshes that are invariant with respect to translation in its each edge). Simplicial TI-meshes can be built only by a uniform decomposition of parallelograms or parallelepipeds into families of equal simplexes. On an arbitrary unstructured mesh the EBR schemes are only exact on linear functions (1-exact) and in practice exhibit the order of accuracy close to second. A high accuracy on TI-meshes allows to gain maximum accuracy within the class of second-order "unstructured" schemes and within the same scheme to combine computations with the close-to-second order of accuracy on unstructured meshes and high-accuracy simulations on structured parts. Numerical results show that the SEBR5 scheme provides higher acuracy than the EBR3 scheme, even on unstructured meshes [35], however a further extension of the reconstruction stencil (for building, for instance, the SEBR7, SEBR9 schemes) practically does not improve the accuracy on unstructured meshes and may negatively affect the scheme robustness. Note that all the edge-based schemes are compatible, i.e. they can be interchanged easilier whenever the mesh quality allows for getting higher accuracy as in the case of TI-meshes.

For problems with discontinuous solutions on unstructured meshes, the very high order methods lose their computational efficiency due to rising of computational cost. Moreover, they usually do not provide better results than the second-order schemes under the existing monotonization techniques. Within this framework, the usage of EBR schemes, in our opinion, can be especially justified. Until recently, however, the potential of these schemes has not been investigated to a full degree.

For solving problems with discontinuities, the quasi-1D edge-based approach seamlessly allows to use the limiting techniques. Thus, a 4-points stencil defined on the edge-based line provides a possibility to implement a higher-accuracy scheme with many known limiters such as minmod, superbee and others for calculating fluxes at the edge midpoints. The paper [34] considers a usage of limiters to keep non-negative values of density and pressure. However, as it is noted above, the use of limiters strongly degrade the accuracy in the regions of smooth solutions. To overcome this problem, in [32,33] devoted to the schemes with quasi-1D reconstructions, the authors suggested to use sensors, however the work on the implementation of the resulting schemes and their analysis has not been completed.

To better treat discontinuities on unstructured meshes, the present paper considers the development of quasi-1D EBR-WENO scheme. For this purpose, 6-points edge-based 1D stencils, which are defined within the SEBR5 scheme, are used for calculating fluxes at the edge midpoints in a way of combination of three different reconstructed values as in the case of the FD WENO5 method. The EBR-WENO5 scheme can be classified as an edge-based extension of finite-difference WENO scheme to unstructured meshes. It combines relatively low costs of edge-based methods, higher accuracy on smooth solutions and damping of non-physical oscillations at discontinuities. We do not consider the EBR-WENO7 and EBR-WENO9 methods due to the properties of underlying SEBR7 and SEBR9 schemes. As for the SEBR schemes, the stencil extension does not improve the accuracy on smooth solutions and may cause additional problems on discontinuities.

## 2. Edge-based schemes

We begin the description of the edge-based schemes from the transport equation

$$\frac{\partial u}{\partial t} + \mathbf{a} \cdot \nabla u = 0 \tag{1}$$

with constant advection velocity $\mathbf{a}$.

Edge-based schemes assume a decomposition of computational domain into duals (cells, or control volumes) which contain only one mesh node. The values of mesh functions are determined in the mesh vertices. In a general form the edge-based schemes are written as follows

$$\frac{du_i}{dt} + \frac{1}{v_i} \sum_{k \in N_1(i)} \mathbf{a} \cdot \mathbf{n}_{ik} h_{ik} = 0, \tag{2}$$

where $N_1(i)$ – a set of nodes that are connected with vertex $i$ by an edge, $\mathbf{n}_{ik}$ – oriented square corresponding to edge $ik$ and directed into the interior of cell $j$, $v_i$ – volume of the cell containing node $i$. In 2D case, these notations are illustrated in Fig. 1 where $\mathbf{n}_{ik} = \mathbf{n}^1_{ik} + \mathbf{n}^2_{ik}$. The conservation property is provided by the identical equations: $\mathbf{n}_{ik} + \mathbf{n}_{ki} = 0$ and $h_{ik} = h_{ki}$. Everywhere in the present paper we use barycentric duals (see [36] or [37]).

The value $h_{ik}$ at the edge midpoint in formulae (2) is calculated taking into account the characteristic direction as

$$h_{ik} = \mathrm{h}(u_{ik}, u_{ki}) = \begin{cases} u_{ik}, & \mathbf{a} \cdot \mathbf{n}_{ik} > 0, \\ u_{ki}, & \mathbf{a} \cdot \mathbf{n}_{ik} < 0, \end{cases} = \begin{cases} \mathcal{R}_{ik}(\{u\}), & \mathbf{a} \cdot \mathbf{n}_{ik} > 0, \\ \mathcal{R}_{ki}(\{u\}), & \mathbf{a} \cdot \mathbf{n}_{ik} < 0. \end{cases} \tag{3}$$