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A numerical model for three-dimensional shallow water flows with sharp gradients over mobile topography

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ABSTRACT

This study aims to develop a three-dimensional (3-D) numerical model for shallow water flows over mobile topography, which is capable of simulating morphological evolution under shock waves, e.g. dam-break flows. The hydrodynamic model solves the three-dimensional shallow water equations (SWEs) using a finite volume method on prismatic cells in σ-coordinates. The morphodynamic model solves an Exner equation consisting of bed-load sediment transportation. Using a relaxation approach, a hyperbolic system is built for hydrodynamic system, which allows for using a Godunov-type central-upwind method to capture the shocks and approximate the numerical fluxes. Consequently, the 3D-SWEs-Exner model proposed in the present study can stably and accurately solve the dam-break flows over mobile beds. A spatially and temporally second-order "prismatic" central-upwind method is used to approximate the numerical fluxes through cell interfaces. The Exner equation is solved using an upwind method. Using spatially linear reconstruction and explicit two-stage Runge–Kutta time discretization, second order accuracy is achieved in space and time. The proposed model can preserve the well-balanced property due to the special discretization of bed-slope source terms. The proposed model is validated by several tests with experimental measurements, and is compared with the simulated results using reported two-dimensional (2-D) models.

1. Introduction

In the past few decades, a number of 2-D numerical models for shallow water (SW) flows with shocks over mobile topography have been developed based on 2-D depth-averaged SWEs. Vásquez [39] have developed a 2-D river morphology model referred as River2D-MOR which has been tested for dam-break flows over mobile bed in [38]. Zhang and Wu have developed a 2-D hydrodynamic and sediment transport model for dam break flows using finite volume method based on quad-tree grids [42]. Xia et al. [41] have developed a 2-D numerical model for dam-break flows over mobile bed using a coupled approach. Evangelista et al. [7] proposed a 2-D model to simulate dam-break waves on movable beds using a multi-stage centered scheme. Swartenbroekx et al. [36] have built a 2-D two-layer shallow water model for dam break flows with significant bed load transport. Liu et al. [22] developed a robust 2-D model for rapidly varying flows over erodible bed with wetting and drying. Canelas et al. [6] proposed a novel two-dimensional depth-averaged simulation tool for highly unsteady discontinuous flows over complex time-evolving geometries. Liu et al. [24] developed a well-balanced 2-D coupled model for dam-break flow over erodible bed. Wei et al. [11] have conducted a 2D hydrodynamic and sediment transport modeling of mega flood due to dam-break. Liu et al. [21] have proposed a two dimensional numerical model for rapidly varying flow over mobile bed, which is suitable for dam-break cases.

There are only a limited number of studies of 3-D shallow water models over erodible beds. Lesser et al. [20] discussed the operation of the DELFT3D module and presented the key features of the formulations used to model the fluid flow and sediment transport over mobile bed, in which the hydrodynamic module solves the 3-D shallow water equations [3,4] based on assumption of hydrostatic status, using the finite difference method. However, the numerical model proposed in [20] cannot guarantee shock-capturing ability, therefore, it may failed in simulations of bed-erosion under unsteady-flows with sharp gradients, e.g., dam-break flows. This prevents the extension of this model to the cases with shocks waves.

Inspired by the study [20], in this study, a 3-D numerical model is developed using the 3-D SWEs and Exner formula. Unlike the previously reported 3-D SW model over mobile bed [20] which may fail in solving sharp gradients, the proposed model can stably

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simulate the bed evolution under dam break waves. This is accomplished by converting the 3-D SWEs into a hyperbolic form using a relaxation approach based on the work [1,23], which allows for the use of shock capturing schemes, e.g. central-upwind method. Consequently, the model is able to simulate dam-break flows without numerical instability. Furthermore, a second order "prismatic" central-upwind method is used to estimate the fluxes through interfaces based on explicit finite volume method. The temporal evolution of topography is estimated by an Exner equation consisting of bed-load transport which is estimated by an upwind method. A stable total variation diminishing two-stage Runge-Kutta (TVDRK2) solver is used in the proposed model to gain second order temporal accuracy. The model is validated by several laboratory tests with experimental data; the proposed 3-D model is compared with several reported 2-D models; and the simulated results using depth-averaged velocities are also compared with those using near bed velocities.

2. Governing equations

2.1. Hydrodynamic model

3-D SWEs with constant density are derived from the 3-D Navier–Stokes equations, after Reynolds-averaging and under the simplifying assumptions of hydrostatic pressure (see [3,4]). Ignoring the horizontal viscous terms and coriolis effect, the 3-D SWEs have the following form:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial \mathrm{w}}{\partial z}=0 \tag{1}$$

$$\frac{\partial u}{\partial t}+\frac{\partial uu}{\partial x}+\frac{\partial vu}{\partial y}+\frac{\partial \mathrm{w}u}{\partial z}=\frac{\partial}{\partial z}\left(\nu_{\mathrm{v}}\frac{\partial u}{\partial z}\right)-g\frac{\partial \eta}{\partial x} \tag{2}$$

$$\frac{\partial v}{\partial t}+\frac{\partial uv}{\partial x}+\frac{\partial vv}{\partial y}+\frac{\partial \mathrm{w}v}{\partial z}=\frac{\partial}{\partial z}\left(\nu_{\mathrm{v}}\frac{\partial v}{\partial z}\right)-g\frac{\partial \eta}{\partial y} \tag{3}$$

where t is the time; x, y and z are the Cartesian coordinates; $u(x, y, z, t)$, $v(x, y, z, t)$ and $\mathrm{w}(x, y, z, t)$ are the velocity components in the x-, y- and z-directions, respectively; g is the gravitational acceleration; $\eta(x, y, t)$ is the water surface elevation; and ν_{v} is the kinematic vertical eddy viscosity. The system (1)–(3) together with either kinematic or dynamic boundary conditions for η constitute a closed system of equations for u, v, w and η.

In order to more accurately define the irregularly bottom boundary, the σ coordinate is adopted in the vertical direction to replace the Cartesian coordinate z using the following transformation [3,4,28]:

$$\sigma=\frac{z-\eta(x,y,t)}{D(x,y,t)}=\frac{z-\eta(x,y,t)}{\eta(x,y,t)-h(x,y,t)}, \tag{4}$$

where σ is the transformed vertical coordinate that varies between -1 and 0, $D(x, y, t)$ is the depth of the water column, and $h(x, y, t)$ is the bed level.

Accordingly, the 3-D SWEs in σ-coordinates can be written in the following form:

$$\frac{\partial \eta}{\partial t}+\frac{\partial (Du)}{\partial x}+\frac{\partial (Dv)}{\partial y}+\frac{\partial \omega}{\partial \sigma}=0 \tag{5}$$

$$\frac{\partial (Du)}{\partial t}+\frac{\partial}{\partial x}\left(Du^2+\frac{g}{2}D^2\right)+\frac{\partial (Dvu)}{\partial y}+\frac{\partial (\omega u)}{\partial \sigma}=\frac{\partial}{\partial \sigma}\left(\frac{\nu_{\mathrm{v}}}{D}\frac{\partial u}{\partial \sigma}\right)-gD\frac{\partial h}{\partial x} \tag{6}$$

$$\frac{\partial (Dv)}{\partial t}+\frac{\partial (Duv)}{\partial x}+\frac{\partial}{\partial y}\left(Dv^2+\frac{g}{2}D^2\right)+\frac{\partial (\omega v)}{\partial \sigma}=\frac{\partial}{\partial \sigma}\left(\frac{\nu_{\mathrm{v}}}{D}\frac{\partial v}{\partial \sigma}\right)-gD\frac{\partial h}{\partial y} \tag{7}$$

where ω is the vertical velocity in the σ-direction, which is related to w by the following relationship:

$$\omega=\mathrm{w}-u\left(\sigma\frac{\partial D}{\partial x}+\frac{\partial \eta}{\partial x}\right)-v\left(\sigma\frac{\partial D}{\partial y}+\frac{\partial \eta}{\partial y}\right)-\left(\sigma\frac{\partial D}{\partial t}+\frac{\partial \eta}{\partial t}\right). \tag{8}$$

We apply the kinematic boundary condition $\omega(x, y, \sigma=-1, t)=0$ and $\omega(x, y, \sigma=0, t)=0$ to close the system (5)–(7). Integrating the continuity Eq. (5) with respect to σ from -1 to 0 using Leibniz's rule gives the following free surface equation:

$$\frac{\partial \eta}{\partial t}+\frac{\partial}{\partial x}\left[\int_{-1}^{0}Du\,d\sigma\right]+\frac{\partial}{\partial y}\left[\int_{-1}^{0}Dv\,d\sigma\right]=0. \tag{9}$$

The vertical eddy viscosity can be estimated by the following model [29] in σ-coordinates:

$$\nu_v=\left(\frac{\ell_{\mathrm{m}}}{D}\right)^2\left[\left(\frac{\partial u}{\partial \sigma}\right)^2+\left(\frac{\partial v}{\partial \sigma}\right)^2\right]^{1/2} \tag{10}$$

where l_m is the mixing length which can be approximated by:

$$\ell_{\mathrm{m}}=\begin{cases}\kappa(\sigma+1)D, & \text{if } \sigma\le -0.75,\\ 0.1D, & \text{otherwise.}\end{cases} \tag{11}$$

where κ=0.41 is the Von Karman constant.

2.2. Morphodynamic model

In the current study, the topography is evaluated by the Exner equation,

$$\frac{\partial h}{\partial t}+\xi\frac{\partial q_{b,x}}{\partial x}+\xi\frac{\partial q_{b,y}}{\partial y}=0 \tag{12}$$

in which ξ is a coefficient equal to $(1-p)^{-1}$; p is the bed porosity; and $q_{b,\,x}$ and $q_{b,\,y}$ are components of bed-load discharge q_b in x and y-directions, respectively.

In this study, the bed-load transport rate is estimated by the Meyer-Peter and Müller (MPM) formula [25],

$$q_b=C_B\sqrt{(s-1)gd^3}\cdot(\theta-\theta_c')^{\frac{3}{2}} \tag{13}$$

in which C_{B} is the bed-load coefficient; s is the specific gravity of sediment given by $s=\rho_s/\rho_f$, where ρ_s is the density of sediment particles and ρ_f is the density of water; d is the mean particle diameter; θ_c' is modified critical bed shear stress, θ_c is the critical bed shear stress that can be estimated by the following formula [35]

$$\theta_c=\frac{0.3}{1+1.2d^*}+0.055(1-e^{-0.02d^*}) \tag{14}$$

where d^* is the dimensionless grain size calculated by

$$d^*=d\left[\frac{g(s-1)}{\nu_f{}^2}\right]^{1/3} \tag{15}$$

where v_f is the kinematic viscosity of fluid, and $v_f=1\times10^{-6}$ m^2/s is used in all the tests in this study. Considering the slope effect, θ_c estimated by (14) can be modified using [34]

$$\theta_c'=\theta_c\frac{\cos\psi\sin\beta+\sqrt{\cos^2\beta\tan^2\phi-\sin^2\psi\sin^2\beta}}{\tan\phi} \tag{16}$$

where $\phi=30^\circ$ is the angle of repose of the bed material (sand) used in the current study, β is the angle of the bed and ψ is the angle between the upslope direction of the sloping bed and the direction of the flow.