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Dirichlet-to-Neumann mappings and finite-differences for anisotropic diffusion

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A B S T R A C T

A general methodology, which consists in deriving two-dimensional finite-difference schemes which involve numerical fluxes based on Dirichlet-to-Neumann maps (or Steklov–Poincaré operators), is first recalled. Then, it is applied to several types of diffusion equations, some being weakly anisotropic, endowed with an external source. Standard finite-difference discretizations are systematically recovered, showing that in absence of any other mechanism, like e.g. convection and/or damping (which bring Bessel and/or Mathieu functions inside that type of numerical fluxes), these well-known schemes achieve a satisfying multi-dimensional character.

1. Preliminaries on “Steklov schemes” for parabolic equations

In the square domain $\Omega = (-1,1)^2$, consider the following Cauchy problem,

$$\partial_t u(t,x,y) - \mathcal{L}(x,y;u) = f(x,y), \qquad u(t=0,\cdot,\cdot) = u^0(\cdot,\cdot) \geq 0, \tag{1}$$

where $\mathcal{L}$ stands for a position-dependent, strictly elliptic differential operator, endowed with Dirichlet boundary data is prescribed on $\partial\Omega$, in accordance with initial data u^0. A uniform Cartesian computational grid on Ω is characterized by constant parameters $\Delta x = \Delta y > 0$: for convenient indexes $i, j \in \mathbb{Z}$, with pointwise approximations read $u^n_{i,j} \simeq u(t^n, x_i, y_j)$ with $t^n = n\Delta t$ (Δt a time-step, $n \in \mathbb{N}$) and $x_i = i\Delta x$, $y_j = j\Delta x$.

1.1. Numerical fluxes being normal derivatives in 1D and 2D

Following the nowadays usual canvas of well-balanced schemes [16], we derive a numerical discretization of (1) which originality is to involve 2D numerical fluxes in which all the terms composing the (discrete) time-derivative would be lumped altogether and treated as a single object. According to [17], after all parameters got “frozen” at each node of the computational grid, a convenient way to build a time-marching scheme for (1) around x_i, y_j consists in computing the residual of C^0 continuity of normal (that is, radial) derivatives of each $\mathcal{L}$-spline touching this point: see Fig. 1.1. Hence, in a more general context, one is led to apply a Steklov–Poincaré (or Dirichlet-to-Neumann) mapping $\mathcal{S}$, see [5,10,24,25], in each “nodal disk”,

$$\mathcal{D} := \mathcal{D}_{i-\frac{1}{2},j-\frac{1}{2}} = \left\{ x,y \text{ such that } |x - x_{i-\frac{1}{2}}|^2 + |y - y_{j-\frac{1}{2}}|^2 \leq R^2 = \frac{\Delta x^2}{2} \right\},$$

in order to express the necessary radial derivatives (which yield outgoing fluxes),

$$\mathcal{S} : H^{\frac{1}{2}}(\partial\mathcal{D}) \to H^{-\frac{1}{2}}(\partial\mathcal{D}) \qquad \bar{\phi} = \phi(R,\cdot) \mapsto \frac{\partial\phi}{\partial r}(R,\cdot), \tag{2}$$

given that $\phi(r,\theta)$ is the “$H^1(\mathcal{D})$-extension of (Dirichlet data) $\bar{\phi}(\theta)$” according to $\mathcal{L}$,

$$\mathcal{L}\phi = f \text{ in } \mathcal{D}, \qquad \phi(R,\theta) = \bar{\phi}(\theta) \text{ on } \partial\mathcal{D}.$$

Definition 1. The “5-points Steklov scheme” for (1) is the time-residual of continuity for the 4 normal derivatives meeting at a given point x_i, y_j:

$$\frac{u^{n+1}_{i,j} - u^n_{i,j}}{\Delta t} - \frac{1}{2R}\Big(\mathcal{S}^{\Delta x}(u^n_{i-1,j}, u^n_{i,j}, u^n_{i,j-1}) + \mathcal{S}^{\Delta x}(u^n_{i,j-1}, u^n_{i,j}, u^n_{i+1,j}) + \mathcal{S}^{\Delta x}(u^n_{i+1,j}, u^n_{i,j}, u^n_{i,j+1}) + \mathcal{S}^{\Delta x}(u^n_{i,j+1}, u^n_{i,j}, u^n_{i-1,j}) \Big) = 0, \tag{3}$$

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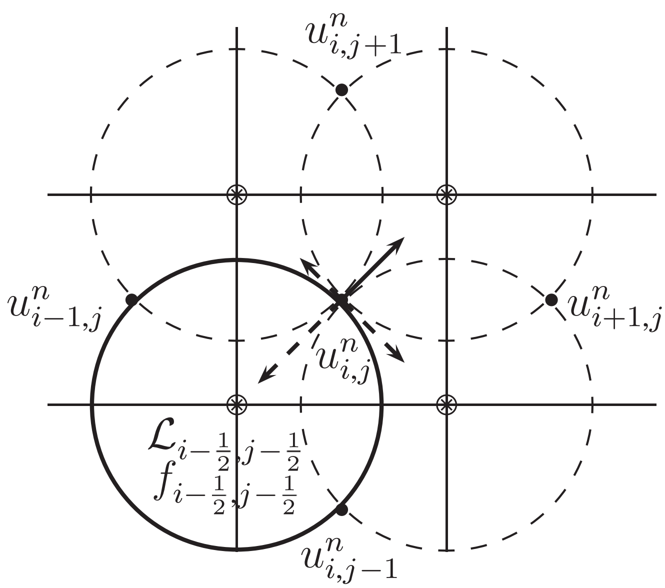

Fig. 1.1. Transmission conditions at $u^n_{i,j}$ given by Steklov-Poincaré operators.

being $\mathcal{S}^{\Delta x}$ an approximate numerical analogue of $\mathcal{S}$ (2) which maps a 3-uple of discrete values belonging to one circle into a normal derivative at x_i, y_j. By extension, the "9-points Steklov scheme" for (1) at a given point x_i, y_j reads:

$$\frac{u^{n+1}_{i,j}-u^n_{i,j}}{\Delta t}-\frac{1}{2R}\Big(\mathcal{S}^{\Delta x}_{i,j}(u^n_{i-1,j},u^n_{i,j},u^n_{i,j-1},u^n_{i-1,j-1}) +\mathcal{S}^{\Delta x}_{i,j}(u^n_{i,j-1},u^n_{i,j},u^n_{i+1,j},u^n_{i+1,j-1})+\mathcal{S}^{\Delta x}_{i,j}(u^n_{i+1,j},u^n_{i,j},u^n_{i,j+1},u^n_{i+1,j+1}) +\mathcal{S}^{\Delta x}_{i,j}(u^n_{i,j+1},u^n_{i,j},u^n_{i-1,j},u^n_{i-1,j+1})\Big)=0, \quad (4)$$

where $\mathcal{S}^{\Delta x}_{i,j}$ is a numerical analogue of (2) mapping a 4-uple of discrete values into a normal derivative at the point x_i, y_j. In this case, it is necessary to indicate the location where the normal derivative is evaluated.

The reason for dividing by $2R=\Delta x\sqrt{2}$ is that normal derivatives are computed at x_i, y_j, but they are considered as attached to the center of each disk: the distance between two centers is $2R$. In the sequel, we shall mostly give the expression of $\mathcal{S}^{\Delta x}(u^n_{i-1,j},u^n_{i,j},u^n_{i,j-1})$, for each choice of the differential operator $\mathcal{L}$, because the 3 others are quite similar.

1.2. Two elementary examples about the heat equation

Hereafter, we present two examples of position-independent problems, for which the former abstract construction (3) can be easily realized:

- Consider first the homogeneous 1D heat equation,

  $$\partial_t u-\partial_{xx}u=0, \qquad \mathcal{L}=-\frac{d^2}{dx^2},$$

  so, as the external source $f\equiv 0$, given discrete data at time t^n, $u^n_j\simeq u(t^n,x_j)$, the $\mathcal{L}$-spline interpolation [26, Ch. 9] is just the piecewise-linear one. Indeed, since

  $$\mathcal{L}v(x)=0, \qquad x\in(x_{j-1},x_j), \qquad v(x_{j-1})=u^n_{j-1},\ v(x_j)=u^n_j,$$

  it comes that

  $$\forall j, \qquad v(x)=u^n_{j-1}+(x-x_{j-1})\frac{u^n_j-u^n_{j-1}}{\Delta x}.$$

  The scheme expresses a time-residual of continuity of normal derivatives,

  $$\forall j,n, \qquad \frac{u^{n+1}_j-u^n_j}{\Delta t}-\frac{1}{\Delta x}\left(\frac{u^n_{j+1}-u^n_j}{\Delta x}-\frac{u^n_j-u^n_{j-1}}{\Delta x}\right)=0,$$

  which is the standard, second-order in space, finite-difference scheme (see [17,18]).
- Yet, consider the 2D homogeneous (rotation-invariant) heat equation,

  $$\partial_t u-\Delta u=0, \qquad \mathcal{L}=\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}. \quad (5)$$

  In any disc centered in $x_{i-\frac{1}{2}},y_{j-\frac{1}{2}}$, as Fig. 1.1 shows, the "$\mathcal{L}$-plate" reads,

  $$\forall r\le R,\ \theta\in(0,2\pi), \quad v_{i-\frac{1}{2},j-\frac{1}{2}}(r,\theta)=a_0+\sum_{k\in\mathbb{N}}(a_k\cos\theta+b_k\sin\theta)r^k. \quad (6)$$

  We truncate that Fourier series at $k<2$: the approximate normal derivative is,

  $$\left.\frac{\partial v}{\partial\vec{n}}\right|_{\partial D}=\left.\frac{\partial v_{i-\frac{1}{2},j-\frac{1}{2}}}{\partial r}\right|_{r=R,\theta=0}=a_1.$$

  Coefficients are given by inverting the 3×3 linear system,

  $$\begin{pmatrix}1 & R & 0\\ 1 & 0 & R\\ 1 & 0 & -R\end{pmatrix}\begin{pmatrix}a_0\\ a_1\\ b_1\end{pmatrix}=\begin{pmatrix}u^n_{i,j}\\ u^n_{i-1,j}\\ u^n_{i,j-1}\end{pmatrix}, \qquad R=\frac{\Delta x}{\sqrt{2}},$$

  so that a discrete "Laplace–Beltrami" normal derivative (see [1]) value is found,

  $$\boxed{a_1=\mathcal{S}^{\Delta x}(u^n_{i-1,j},u^n_{i,j},u^n_{i,j-1})=\frac{u^n_{i-1,j}-2u^n_{i,j}-u^n_{i,j-1}}{2R},} \quad (7)$$

  and once again, this corresponds to a standard 5-points finite-difference scheme, as can be easily checked by assembling all the expressions appearing in (3):

  $$\frac{u^{n+1}_{i,j}-u^n_{i,j}}{\Delta t}-\frac{1}{2R}\Big(\frac{u^n_{i-1,j}-2u^n_{i,j}-u^n_{i,j-1}}{2R}+\frac{u^n_{i,j-1}-2u^n_{i,j}-u^n_{i+1,j}}{2R} +\frac{u^n_{i+1,j}-2u^n_{i,j}-u^n_{i,j+1}}{2R}+\frac{u^n_{i,j+1}-2u^n_{i,j}-u^n_{i-1,j}}{2R}\Big)=0, \quad (8)$$

  and simplifying because $4R^2=2\Delta x^2$.

1.3. Outline and organization of the paper

A contemporary trend in Numerical Analysis lies in designing "truly multi-dimensional discretizations" of PDE's in two (or even, three) dimensions, by going beyond classical Taylor expansions (finite differences) or tensorial products of polynomials (finite elements). A paradigm of these concerns appears in [12] where the linear 2D wave equation is discretized by relying on a reformulation of Kirchhoff's exact solution as a time-marching, Lax–Wendroff type, numerical scheme (see also [14] for an application of these ideas to uniform Cartesian grids). Meanwhile, for parabolic equations, a closely related framework was proposed in [18], which consists in designing multi-dimensional time-marching finite-difference schemes by applying Dirichlet-to-Neumann (DtN) maps in local domains centered around each node of the computational grid: this is the idea that will be examined hereafter in more detail, especially with the aim of verifying that, when the equation is simple enough, such (apparently unusual) algorithms reduce to well-known, standard finite-differences.