Computers and Fluids 156 (2017) 456–469

Contents lists available at ScienceDirect

Computers and Fluids

journal homepage: www.elsevier.com/locate/compfluid

Benchmark solutions

Stochastic representation of the Reynolds transport theorem: Revisiting large-scale modeling

S. Kadri Harouna[a], E. Mémin[b,*]

[a] Laboratoire Mathématiques, Image et Applications (MIA), Université de La Rochelle, Avenue Michel Crépeau 17042, La Rochelle France
[b] INRIA Rennes, Campus Universitaire de Beaulieu, 35042 Rennes cedex, France

ARTICLE INFO

Article history:
Received 6 August 2016
Revised 8 May 2017
Accepted 4 August 2017
Available online 8 August 2017

Keywords:
Large-scale fluid flow dynamics
Stochastic transport
Subgrid model
Turbulence
Taylor-Green flow

ABSTRACT

We explore the potential of a formulation of the Navier-Stokes equations incorporating a random description of the small-scale velocity component. This model, established from a version of the Reynolds transport theorem adapted to a stochastic representation of the flow, gives rise to a large-scale description of the flow dynamics in which emerges an anisotropic subgrid tensor, reminiscent to the Reynolds stress tensor, together with a drift correction due to an inhomogeneous turbulence. The corresponding subgrid model, which depends on the small scales velocity variance, generalizes the Boussinesq eddy viscosity assumption. However, it is not anymore obtained from an analogy with molecular dissipation but ensues rigorously from the random modeling of the flow. This principle allows us to propose several subgrid models defined directly on the resolved flow component. We assess and compare numerically those models on a standard Green-Taylor vortex flow at Reynolds numbers $Re = 1600$, $Re = 3000$ and $Re = 5000$. The numerical simulations, carried out with an accurate divergence-free scheme, outperform classical large-eddies formulations and provides a simple demonstration of the pertinence of the proposed large-scale modeling.

1. Introduction

The large-scale modeling of fluid flow dynamics remains nowadays a major research issue in fluid mechanics or in geophysics despite an enormous research effort since the first investigations on the subject 150 years ago [6]. The research themes behind this topic cover fundamental issues such as turbulence modeling and the analysis of fully developed turbulent flows, but also more applicative research problems related to the definition of practical numerical methods for the simulation of complex flows. In this latter case the difficulty consists in setting up a reliable modeling of the large-scale dynamics in which the contribution of unresolved small-scale processes are explicitly taken into account. For the Navier-Stokes equations, the problem is all the more difficult that the spatial and temporal scales are tightly interacting together.

The neglected processes include, among others things, the action of the unresolved motion scales, complex partially-known forcing, an incomplete knowledge of the boundary conditions and eventual numerical artifacts. Such unresolved processes must be properly taken into account to describe accurately the energy transfers and to construct stable numerical simulations. In real world situations, the complexity of the involved phenomenon prevents the use of an accurate – but inescapably expensive – deterministic modeling. We advocate instead the use of a stochastic modeling.

Within this prospect, we aim at describing the missing contributions through random fields encoding a flow component only in a probability distribution sense. Those variables correspond to the discrepancies or errors between the dynamical model and the actual dynamics. Their modeling is of the utmost importance in geophysics, either for data assimilation or forecasting issues. In both cases, an accurate modeling of the flow errors dynamics enables to maintain an ensemble of flow configurations with a sufficient but also meaningful diversity.

Small-scale processes are responsible both for an energy dissipation but also for local backscattering of energy [56]. The introduction of random variables in the flow dynamics has been considered by several authors, as it constitutes an appealing mechanism for the phenomenological modeling of intermittent processes associated to the inverse energy cascade [39,45,65]. Recently those models have regained a great interest for the modeling of geophysical flows [11,27,43,44,67] in climate sciences (see also the thematic issue [53] or the review [19]).

Numerous turbulence models proposed in the context of Large Eddies Simulations (LES) and Reynolds Average Simulations (RANS)

* Corresponding author.
E-mail addresses: etienne.memin@inria.fr, Etienne.Memin@inria.fr (E. Mémin).

http://dx.doi.org/10.1016/j.compfluid.2017.08.017
0045-7930/

introduce *de facto* an eddy viscosity assumption to model the energy dissipation due to unresolved scales [[15,40,47], [63], [68]]. This concept dates back to the work of Boussinesq [6] and Prandtl [59]. It relies on the hypothesis that the energy transfer from the resolved scales to the subgrid scales can be described in a similar way to the molecular viscosity mechanism. It is therefore not at all related to any uncertainty or error quantities. In models dealing explicitly with a statistical modeling of the turbulent fluctuations there is thus some incoherency in representing directly the dissipative mechanism attached to random terms through an eddy viscosity assumption. In this work we will not make use of such an hypothesis. Instead, we will rely on a general diffusion expression that emerges naturally from our formalism.

This subgrid model is properly derived from a general Lagrangian stochastic model of the fluid motion in which the fluid parcels displacement is decomposed in two components: a smooth differentiable (possibly random) function and a random field, uncorrelated in time but correlated in space. Such a decomposition consists in separating or "filtering" a rough velocity in a smooth slow time-scale component and a fast oscillating velocity field representing the unresolved flow. Though there is, in general, no sharp time-scale separation in turbulent flows, the resolved velocity can be interpreted as a temporally coarse-grained component whereas the time-uncorrelated component stands for the small time-scale unresolved velocity. As a temporal smoothing imposes implicitly a spacial smoothing, this separation can be thus interpreted in terms of a LES filtering technique. Yet, the corresponding Eulerian formulation does not ensue from a filtering procedure. It is thus not prone to errors associated to the violation of the commutation assumption between the filter and the spatial derivatives [23,24]. Besides, those equations introduce an effective advection related to the small-scale velocity inhomogeneity. This modified advection, empirically introduced in Langevin models of particle dispersion [42], corresponds exactly to a phenomenon, termed *turbophoresis*, related to the migration of inertial particles in regions of lower turbulent diffusivity [66].

The large-scale representation of the Navier-Stokes equations on which we rely in this study are built from a stochastic version of the Reynolds transport theorem [46]. This modified Reynolds transport theorem, which constitutes here the cornerstone of our large-scale dynamics representation, is presented in the following section. General invariance properties of the corresponding large-scale dynamics such as scale and Galilean invariances are detailed in a comprehensive appendix. In Section 3 several novel subgrid tensors will be devised and compared on a standard Green-Taylor vortex flow [70]. We will show that all the proposed schemes outperform the usual dynamic Smagorinsky subgrid formulation [21,22,41,68].

2. Stochastic modeling of fluid flow dynamics

Numerous methodological choices can be envisaged to devise stochastic representations of the Navier-Stokes equations. The simplest method considers additional random forcing to the dynamics. This is the choice that has been the most often performed since the work of Benssoussan [3]. Another choice, in the wake of Kraichnan's work [34], consists in closing the large-scale flow representation in the Fourier space by relying on a Langevin equation [30,36,38]. Obviously the frontiers between those two methodologies are sometimes fuzzy, and numerous works rely on both strategies in order to devise the shape that should take the random variables evolution law [30,65]. Lagrangian stochastic models based on Langevin equations have been also intensively used for turbulent dispersion [64] or in probability density function (PDF) modeling of turbulent flows [28,58]. Those Lagrangian models, which require to model the drift and diffusion functions, lead to very attractive particle based representations of complex flows [48,57]. They are nevertheless not adapted to global large-scale Eulerian representations of the flow dynamics.

In this work, we will rely on a different framework in specifying the stochastic nature of the velocity from the very beginning as proposed in [29,46]. The basic idea is built on the assumption that the Lagrangian fluid particles displacement results from a smooth velocity component and a highly oscillating stochastic velocity component uncorrelated in time,

$$\boldsymbol{X}_t = \boldsymbol{X}_{t_0} + \int_{t_0}^{t} \boldsymbol{w}(\boldsymbol{X}_s, s)\mathrm{d}s + \int_{t_0}^{t} \boldsymbol{\sigma}(\boldsymbol{X}_s, s)\mathrm{d}\boldsymbol{B}_s, \tag{1}$$

with the velocity components:

$$\boldsymbol{U}(\boldsymbol{X}_t, t) = \boldsymbol{w}(\boldsymbol{X}_t, t) + \dot{\boldsymbol{W}}(\boldsymbol{X}_t, t). \tag{2}$$

In this decomposition the first right-hand term is a smooth function of time associated to the large-scale velocity component. The second term stands for the small-scale velocity field. It is a white noise velocity component defined from the (formal) time-derivative of the random field: $\dot{\boldsymbol{W}}(\boldsymbol{X}_t, t) = \boldsymbol{\sigma}(\boldsymbol{X}_t, t)\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{B}'_t$. This random field is a three-dimensional centered Wiener process; it is thus uncorrelated in time but can be anisotropic and inhomogeneous in space. Since we focus in this study only on incompressible flows, the small-scale component is defined as a divergence-free random field; it is hence associated to a divergence-free diffusion tensor:

$$\nabla \cdot \boldsymbol{\sigma} = 0. \tag{3}$$

Analogously to the standard deterministic case, the derivation procedure from the physical conservation laws of the Navier-Stokes equations is based primarily on the Reynolds transport theorem (RTT).

2.1. Stochastic reynolds transport theorem

The RTT provides the expression of the rate of change of a scalar function, q, within a material volume, $\mathcal{V}(t)$. For a stochastic flow (2) with an incompressible small-scale velocity component ($\nabla \cdot \boldsymbol{\sigma} = 0$), this expression derived in [46,60], is given by:

$$\mathrm{d}\int_{\mathcal{V}(t)} q\mathrm{d}\boldsymbol{x} = \int_{\mathcal{V}(t)} \Bigg(\mathrm{d}_t q + \Bigg[\nabla \cdot \Big(q \underbrace{\Big(\boldsymbol{w} - \frac{1}{2}\nabla \cdot \boldsymbol{a} \Big)}_{\widetilde{\boldsymbol{w}}} \Big) - \frac{1}{2}\sum_{i,j=1}^{d} \partial_{x_i}(a_{ij}\partial_{x_j} q) \Bigg]\mathrm{d}t + \nabla q \cdot \boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}'_t \Bigg)\mathrm{d}\boldsymbol{x}. \tag{4}$$

This modified RTT involves the time increment of the random scalar quantity q (the differential of q at a fixed point) instead of the time derivative. A diffusion operator emerges also naturally. For clarity's sake, this term is designated as "subgrid stress tensor" following the protocols of large eddies simulation (LES). However, its construction is quite different. It is not based on Boussinesq's eddy viscosity assumption nor on any structural turbulence models [[63]] but arises directly from stochastic calculus rules. It expresses the mixing process exerted on the scalar quantity by the fast oscillating velocity component. This diffusion term is directly related to the small-scale component through the *variance tensor*, $\boldsymbol{a}$, defined from the diagonal of the small-scale velocity covariance:

$$\boldsymbol{a}(\boldsymbol{x}, t)\delta(t - t')\mathrm{d}t = \mathbb{E}((\boldsymbol{\sigma}(\boldsymbol{x}, t)\mathrm{d}\boldsymbol{B}'_t)(\boldsymbol{\sigma}(\boldsymbol{x}, t')\mathrm{d}\boldsymbol{B}'_{t'})^T),$$

it can be checked that the variance tensor corresponds to an eddy viscosity term (with units in m^2s^{-1}). This term plays thus a role similar to the eddy viscosity models introduced in classical large scale representations [2,20,41,68] or to the spectral vanishing viscosity [33,54,69]. It is also akin to numerical regularization models