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The effect of surface tension on free surface flow induced by a point sink in a fluid of finite depth

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ABSTRACT

Solutions are presented to the problem of steady, axisymmetric flow of an inviscid fluid into a point sink. The fluid is of finite depth and has a free surface. Two numerical schemes, a spectral method and an integral equation approach, are implemented to confirm results for the maximum-flow-rate steady solution for each configuration. The effects of surface tension and sink depth are included and constitute the new component of the work. Surface tension has the effect of increasing the maximum flow rate at which steady-state solutions can exist.

1. Introduction

The problem of steady flow due to a single, motionless sink beneath a free surface has proven deceptively difficult. More accurately, while it is relatively easy to obtain numerical solutions to this problem, the limiting parameters for which steady flows exist have proven difficult to find with confidence. While it is generally accepted that as the flow rate increases there comes a point beyond which steady solutions no longer exist, this critical value has had multiple proposed values in the literature using similar numerical methods. The current work uses two completely different numerical approaches to resolve the critical values at which steady solutions cease to exist for the flow into a point sink above a horizontal base. The agreement of these two different approaches is central to the conclusions drawn about the solutions. Surface tension is included in the work, both to gauge its influence on the flows and for its stabilizing effect on both the flow and the numerical schemes.

While the results are of mathematical interest as a fundamental study of free surface hydrodynamics, the problem is also relevant to the withdrawal of fluid from water storage reservoirs and other confined water bodies [15,16]. Fluid withdrawn from reservoirs tends to flow in layers due to the density stratification inherent in all reservoirs in temperate climatic zones. This vertical stratification often consists of constant density regions and regions with approximately linear density variation due to either temperature or salinity. An understanding of the process of selective withdrawal is important in delivering suitable water quality for urban and agricultural supply.

Peregrine [20] proposed the analogous problem in two dimensions (with a line sink) as a study that might assist in understanding wave-breaking, and while this has proven not to be the case for steady flow, some wave breaking behaviour has been observed in the unsteady version in which the sink is turned on in a fluid at rest [22]. Regardless, the steady problem with a line sink has provided some very interesting behaviour and due to the (relative) ease of computation and the availability of complex variable methods [2,4,8,20,21,25,26] there has been much work on this case. Surface tension was considered in [4], and withdrawal in the presence of a background flow by Holmes and Hocking [14]. In both cases non-uniqueness was found in the solution space. Two kinds of steady solution were obtained for flow from a single layer fluid with a free surface, one involving a stagnation point on the surface and another involving a cusp above the sink [21,25,26]. Hocking [9] and Hocking and Forbes [10] showed that the cusp solutions correspond to the situation in which the free surface is pulled down directly into the sink if the withdrawal rate is increased beyond this value. Thus, if there is another fluid above this layer, this flow corresponds to the transition to a two-layer flow in which fluid from both layers flow out through the sink. This was found to be true in both an unconfined fluid and a fluid of finite depth. Numerical calculations of the unsteady flow indicate that this critical drawdown flow is related to the maximum steady flow, but the

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actual drawdown of the interface between two layers occurs at a flow rate that depends on the flow history [22–24].

This considerable progress in the two dimensional case has not really been matched in the problem of flow due to a point sink. Such flows were considered experimentally by the authors in Refs. [7,17,18] and others, and later in a full simulation by Zhou and Graebel [29] and Xue and Yue [28]. Miloh and Tyvand [19] considered a small time expansion to look for critical drawdown values. No solutions with the equivalent of a cusp shape have been found, except over a small range of parameters [6].

The first computations of steady solutions for a point sink with a stagnation point were performed by Forbes and Hocking [3] and Vanden-Broeck and Keller [27] on the case of a semi-infinite fluid. These authors solved the same equations but using different numerical approaches. The problem can be defined in terms of a Froude number $F_S = \sqrt{m^2/(gH_S^5)}$, where m is the sink strength, g is gravitational acceleration and H_S is the depth of the sink. While the former found a limiting value of $F_S \approx 6.4$, the latter obtained values close to $F_S = 5.4$ (with a train of decaying upstream waves) using essentially the same integral equation approach. The limiting solutions in both cases appeared to have the same physical characteristic of a stagnation ring on the free surface some small distance from the central surface stagnation point. Solutions were computed using integral equation techniques pushing the limits of computer power of the time. Similar discrepancies in the critical values appeared when a flow with the sink on a horizontal, impermeable base were computed by Hocking et al. [11] (using the numerical approach of [27]) who found $F_S \approx 3.24$, while [5] obtained a much lower value of $F_S \approx 1.5$ using a fundamental singularity, Galerkin technique. The former contained the familiar stagnation ring limiting solution, but the latter did not. Experiments and full numerical simulations in various geometries produced values for limiting single layer flows ranging from $F_S \approx 1.6$ [7] to $F_S \approx 3$ [29] although these may not be directly related to limiting steady-state solutions with a central stagnation point.

A recent, more thorough analysis of the integral equation method was given in [12], and it was shown that the limiting steady solutions occur at much lower values of flow rate than initial calculations suggest. Surface tension was included in [13] and was found to have a regularizing effect on both the solutions and the existence space, so that much higher flow rates could be obtained with significant surface tension included. The “errors” appear to be due to inappropriate truncation [3] and lack of convergence of the numerical scheme as grid spacing was decreased [27], but in both cases this was not obvious with the computational capacity available at the time.

Here we consider the problem with a point sink situated at an arbitrary depth in a fluid of finite depth and include the effects of surface tension, see Fig. 1. Two different numerical schemes were used and found to give matching solutions for all parameter values. Again, the effect of surface tension is to regularize the flow. By taking the limit as the surface tension approaches zero we were able to confirm the limiting values for zero surface tension.

2. Problem formulation

Consider the steady, irrotational, axisymmetric flow of an inviscid, incompressible fluid beneath a free surface. The flow is driven by a point sink of strength m situated at a depth H_S beneath the undisturbed level of the free surface and above a flat impermeable boundary at depth D. Under these assumptions the problem can be formulated in terms of a velocity potential $\phi(r, z)$, where r is a radial coordinate centred on the location of the point sink and z is the vertical coordinate with $z = 0$ corresponding to the level of the free surface if there is no flow. Thus the velocity can be obtained as $\nabla\phi = (u, w)$, where u is the radial component and w is the vertical component. The free surface is subject to surface tension, T.

Nondimensionalising the potential and length with respect to (m/D) and D, respectively, where the quantity m is the strength of the point sink, the problem is to solve

$$\nabla^2\phi = 0, \quad -1 < z < \eta(r), \quad (r, z) \neq (0, -h_S), \tag{1}$$

subject to the dynamic condition obtained from setting pressure to the atmospheric value on the free surface in the Bernoulli equation, i.e.

$$\eta + \frac{F_D^2}{2}(u^2 + w^2) - \frac{\beta\left(r\eta_{rr} + \eta_r(1 + \eta_r^2)\right)}{r[1 + \eta_r^2]^{3/2}} = 0 \quad \text{on} \quad z = \eta(r) \tag{2}$$

with a kinematic condition that no flow can occur through the surface in steady flow given by

$$\nabla\phi \cdot \mathbf{n} = \phi_r\eta_r - \phi_z = 0 \quad \text{on} \quad z = \eta(r), \tag{3}$$

where $\mathbf{n}$ is the normal to the free surface, and a condition that there can be no flow through the impermeable base beneath the layer of fluid,

$$\phi_z = 0 \quad \text{on} \quad z = -1. \tag{4}$$

These equations include the main parameters that control this flow; the Froude number, the sink depth and the surface tension

$$F_D = \left(\frac{m^2}{gD^5}\right)^{1/2}, \quad h_S = H_S/D, \quad \beta = \frac{T}{gD^2} \tag{5}$$

in which g is gravitational acceleration. In most cases the Froude number can be thought of as an effective flow rate. We can define a second Froude number that is based on the depth of the sink rather than the depth of the fluid as

$$F_S = \left(\frac{m^2}{gH_S^5}\right)^{1/2}. \tag{6}$$

The value of F_S is related to F_D via the relation $F_D = h_S^{5/2}F_S$, where h_S is the nondimensional sink depth, and is useful for comparison with values computed in an unbounded fluid, for which $F_D \to 0$ as $D \to \infty$.

In the limit as we approach the point sink at $(r, z) = (0, -h_S)$ the velocity potential should take the form

$$\Phi_S \to \frac{1}{4\pi\sqrt{r^2 + (z + h_S)^2}} \tag{7}$$

which corresponds to a total flux into the sink of $Q = 4\pi$. A change of sign reverses the flow direction from a sink flow to a source flow. However, in the case of steady flow, the quadratic nature of the velocity term in the dynamic condition (2) means that steady solutions are valid for both a source and a sink.

3. Rigid-lid solution

It is of interest to compute a solution that is valid for small flow rates that result in a small disturbance to the free surface. In essence we can compute the flow due to a point sink confined in a horizontal duct. An expansion about the flow along the top of the duct is used to approximate the shape of the free surface. The linearized problem is thus to solve Laplace’s equation in the region $-1 < z < 0$ subject to the linearized kinematic conditions of $\phi_z = 0$ on $z = 0, -1$. The dynamic condition (2) can then be used to estimate the shape of the free surface by expanding about $z = 0$. Following the usual procedure of allowing $\phi = \phi_0 + \phi_1 + \cdots$ and