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The emergence of fast oscillations in a reduced primitive equation model and its implications for closure theories☆

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ABSTRACT

The problem of emergence of fast gravity-wave oscillations in rotating, stratified flow is reconsidered. Fast inertia-gravity oscillations have long been considered an impediment to initialization of weather forecasts, and the concept of a "slow manifold" evolution, with no fast oscillations, has been hypothesized. It is shown on a reduced Primitive Equation model introduced by Lorenz in 1980 that fast oscillations are absent over a finite interval in Rossby number but they can develop brutally once a critical Rossby number is crossed, in contradistinction with fast oscillations emerging according to an exponential smallness scenario such as reported in previous studies, including some others by Lorenz. The consequences of this dynamical transition on the closure problem based on slow variables is also discussed. In that respect, a novel variational perspective on the closure problem exploiting manifolds is introduced. This framework allows for a unification of previous concepts such as the slow manifold or other concepts of "fuzzy" manifold. It allows furthermore for a rigorous identification of an optimal limiting object for the averaging of fast oscillations, namely the optimal parameterizing manifold (PM). It is shown through detailed numerical computations and rigorous error estimates that the manifold underlying the nonlinear Balance Equations provides a very good approximation of this optimal PM even somewhat beyond the emergence of fast and energetic oscillations.

1. Introduction

The concept of a "slow manifold" was presented in a didactic paper by Leith [37] in an attempt to filter out, on an analytical basis, the fast gravity waves for the initialization of the Primitive Equations (PE) of the atmosphere. The motivation was that small errors in a "proper balance" between the fast time-scale motion associated with gravity waves and slower motions such as associated with the Rossby waves, lead typically to an abnormal evolution of gravity waves, which in turn can cause appreciable deviations of weather forecasts. This filtering approach has a long history in forecast initialization, e.g. [3,43].

To provide a remedy to this initialization problem, Leith proposed that a "proper balance" between fast and slow motion may be postulated to exist, and, using the language of dynamical system theory, it was thought of as a manifold in the phase space of the PE consisting of orbits for which gravity waves motion is absent. An iteration scheme was then developed to find from the observed state in phase space a corresponding initial state on such a "slow" manifold, so that weather forecasts with these initial states can be accurate on the same time scales as those of Rossby waves. In Leith's treatment the filtering was equivalent to the Quasigeostrophic approximation for asymptotically small Rossby number, V/fL (V a typical horizontal velocity, f the Coriolis frequency, and L a horizontal length). Solutions to the Quasigeostrophic model remain slow for all time.

This idea was appealing for dealing with this filtering problem, but uncertainty in the definition of a slow manifold for finite Rossby number has led to a proliferation of different schemes, on one hand, and to the question of whether a precise definition can be provided at all on the other hand, i.e., whether a slow invariant manifold even exists at finite Rossby number.

The latter question is especially interesting from a theoretical point of view. Lorenz [41] was probably the first to address in at-

☆ This article is dedicated to Chuck, a longtime colleague of the 3rd author, and it is presented as fruitful conversations across our generations.

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mospheric sciences the problem of definition and existence of a slow manifold as a dynamical system object, although the concept was analyzed by mathematicians prior to that work [20,21,56]. In that respect, he introduced a further simplified version of his truncated, nine-dimensional PE model derived originally in [40] to reduce it to a five dimensional system of ordinary differential equations (ODEs). He then identified the variables representing gravity waves as the ones which can exhibit fast oscillations, and defined the slow manifold as an invariant manifold in the five dimensional phase space for which fast oscillations never develop. In a subsequent work, Lorenz and Krishnamurthy [38] after introducing forcing and damping in the 5-variable model of [41], identified an orbit which by construction has to lie on the slow manifold. They followed its evolution numerically to show that sooner or later fast oscillations developed, thereby implying that a slow manifold according to their definition did not exist for the model.

By relying on quadratic integral of motions, it was shown in [4] that the 5-variable model of [41] reduces to the following slow-fast system of four equations:

$$\begin{aligned} \dot{\theta} &= w - \epsilon b y, \\ \dot{w} &= -\sin(\theta), \\ \epsilon \dot{x} &= -y, \\ \epsilon \dot{y} &= x + b\sin(\theta). \end{aligned} \tag{1.1}$$

In this form, the Lorenz–Krishnamurthy (LK) system (without dissipation and forcing terms) can be understood as describing the dynamics of a slow nonlinear pendulum (w, θ), with angle θ from the vertical, coupled in some way with a harmonic oscillator that can be thought as a stiff spring with constant ϵ^{-1} and of extension (x, y).

By a delicate usage of tools from the geometric singular perturbation theory [32] to "blow up" the region near the singularity (of a saddle-center type)[1] at the origin, it was rigorously shown in [4] that the time evolution of initial data lying on the (homoclinic) orbit considered in [38] will invariably develop fast oscillations in the course of time. This result provided a partial answer to the question raised in [38] about the existence of a slow manifold, at least in the conservative case.

Nevertheless, the outcome of such a study was seemingly in contradiction with those of [30], which show, by relying essentially on a local normal form analysis, that for the (dissipative) LK system, a slow manifold exists. As noted by Lorenz himself in [42], again what one means by "slow manifold" does matter. In [30], the existence of such a manifold was only local in the phase space, which did not exclude thus the emergence of fast oscillations as one leaves the neighborhood of the relevant portion of the phase space, here near the Hadley point $(0, F, 0, 0, 0)$.[2] Actually, the authors of [15] proved that a global manifold can be identified, but that this manifold is not void of fast oscillations and thus is not slow in the language of dynamical system theory.

The implications of the results of [15] combined with the original numerical results of [38], advocated thus an interesting physical mechanism for the spontaneous generation of inertia-gravity waves. Lorenz and Krishnamurthy used numerical solutions to show in the low-Rossby-number, Quasigeostrophic regime that the amplitude of the inertia-gravity waves that are generated is actually exponentially small, i.e. proportional to $\exp(-\alpha/\epsilon)$, where $\epsilon <<1$ is the relevant small parameter and $\alpha > 0$ is a structural constant. The generation of exponentially small inertia-gravity oscillations takes place for $t > 0$, whereas the solutions are well balanced for $t \to -\infty$.

By means of elegant exponential-asymptotic techniques, Vanneste in [59] provided an estimate for the amplitude of the fast inertia-gravity oscillations that are generated spontaneously, through what is known as of the crossing of Stokes lines as time evolves, i.e. the crossing of particular time instants corresponding to the real part of poles close to the real (time) axis, in the meromorphic extension of the solutions (in complex time). These analytic results showed thus an exponentially small "fuzziness" scenario (in Rossby number) to hold for the LK system; exponential smallness then has been argued to hold for more realistic flows by several complementary studies or experiments; e.g. [22,51,60,61,63,64].

Going back to the original reduced PE model of Lorenz [40], we show on a rescaled version (described in Section 2.2) that while the emergence of small-amplitude fast oscillations is still synonymous of the breakdown of (exact) slaving principles, a sharp dynamical transition occurs as a parameter ϵ, which can be identified with the Rossby number, crosses a critical value ϵ_*. Such a sudden transition was pointed out in [62]. We conduct in this work a more detailed examination of this transition with in particular smaller time steps and a higher-order time-stepping scheme than used in [62]. This transition corresponds to the emergence of fast gravity waves that can contain a significant fraction of the energy (up to $\sim$ 40%) as time evolves and that may either populate transient behaviors of various lengths or persist in an intermittent way as both time flows and ϵ varies beyond ϵ_*; see Section 2.3. Although the mathematical characterization of this transition is an interesting question *per se*, we focus in this article on the consequences of such a critical transition on the closure problem for the slow rotational variables. For that purpose we revisit the Balance Equations (BE) [27] within the framework of *parametrizing manifolds (PMs)* introduced in [9,12] for different but related parameterization objectives.

As shown in Sections 3 and 4 below, the PM approach introduces a novel variational perspective on the closure problem exploiting manifolds which allows us to unify within a natural framework previous concepts such as the slow manifold [37] or other notions of *approximate inertial manifolds* [17,57,58], as well as the "fuzzy manifold" [41,65,68] or "quasi manifold" [22]. This variational approach can even be made rigorous as shown in Appendix A. Theorem A.1, proved therein, shows indeed that an optimal PM always exists and that it is the optimal manifold that averages out the fast oscillations, i.e. the best fuzzy manifold one can ever hope for in a certain sense. Detailed numerical computations and rigorous error estimates (see Proposition 3.1) as well as comparison with other natural manifolds such as that associated with the Quasigeostrophic (QG) balance (see Section 4.2), show that the manifold underlying the BE provides a very good approximation of this optimal PM even beyond the criticality, when the fast gravity waves contain a large fraction of the energy.

The framework introduced in this article allows us furthermore to relate the optimal PM to another key object, the *slow conditional expectation*. As explained in Section 4.1 below, the slow conditional expectation provides the best vector field of the space of slow variables that approximates the PE dynamics, and it can be easily derived from the optimal PM (and thus the BE in practice); see (4.7) below. This slow conditional expectation (and thus the optimal PM) becomes however insufficient for closing with only the slow variables, i.e. for ϵ-values beyond ϵ_* for which an explosion of energetic fast oscillations occurs, as explained in Section 4.3. It is shown then that corrective terms are needed in such a situation. These terms take the form of integral terms accounting for the cross-interactions between the slow and fast variables that the

[1] This point corresponds to the unstable equilibrium of the pendulum and the neutral equilibrium of the harmonic oscillator.

[2] This point is an hyperbolic equilibrium of the LK system, a property that allows for the application of the standard Hartman-Grobman theory which can be furthermore combined with the Siegel's linearization theory [1] to infer rigorously to the existence of a local slow manifold; see [15].