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Building a Godunov-type numerical scheme for a model of two-phase flows

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ABSTRACT

Our aim in the paper is to build a Godunov-type numerical scheme for an isentropic model of two-phase flows. First, computational Riemann solvers together with computing algorithms in subsonic and supersonic regions are presented. Then, exact solutions of local Riemann problems are employed to build a Godunov-type scheme. The scheme is shown to be well-balanced in the sense that it can capture exactly stationary waves in both phases. Numerical tests show that the Godunov-type scheme possesses a good accuracy in the subsonic region as well as supersonic regions, where approximate solutions are convergent to the exact solution.

1. Introduction

In this paper we will construct a Godunov-type numerical scheme for the following isentropic model of two-phase flows

$$
\begin{aligned}
&\partial_t(\alpha_g\rho_g) + \partial_x(\alpha_g\rho_g u_g) = 0,\\
&\partial_t(\alpha_g\rho_g u_g) + \partial_x\big(\alpha_g(\rho_g u_g^2 + p_g)\big) = p_g\partial_x\alpha_g,\\
&\partial_t(\alpha_s\rho_s) + \partial_x(\alpha_s\rho_s u_s) = 0,\\
&\partial_t(\alpha_s\rho_s u_s) + \partial_x\big(\alpha_s(\rho_s u_s^2 + p_s)\big) = -p_g\partial_x\alpha_g,\\
&\partial_t\rho_s + \partial_x(\rho_s u_s) = 0, \quad x\in\mathbb{R}, \quad t>0,
\end{aligned}
\tag{1.1}
$$

where α_g, ρ_g, u_g, p_g denote the volume fraction, density, velocity, and pressure in the first phase of flow, called *Phase I*; and α_s, ρ_s, u_s, p_s denote the volume fraction, density, velocity, and pressure in the second phase of flow, called *Phase II*. The system (1.1) is obtained from the modeling of deflagration-to-detonation transition in granular materials, see [7,8], where the flows are assumed to be nonreactive and isentropic. The volume fractions satisfy

$$\alpha_g + \alpha_s = 1. \tag{1.2}$$

The first and the second equations of (1.1) describe the balance of mass and momentum in Phase I, while the third and the four equations of (1.1) describe the balance of mass and momentum in Phase II; the fifth equation of (1.1), called the *compaction dynamics equation*, represents the evolution of the volume fractions. Observe that the third and the fifth equations of (1.1) and the Eq. (1.2) yield

$$\partial_t\alpha_g + u_s\partial_x\alpha_g = 0. \tag{1.3}$$

The model (1.1) contains nonconservative terms, which reflect the exchanges between the two phases. We will see that it can be written as a hyperbolic system of balance laws in nonconservative form. Theoretically, weak solutions of this kind of system can be understood in the sense of *nonconservative product*, see [17]. Nonconservative terms in multi-phase flow models, and more generally, in nonconservative systems of balance laws, often derive very hard obstacle for numerical approximations of the solutions. In particular, the errors may not tend to zero as the mesh sizes go to zero for standard numerical schemes. Therefore, theoretical study as well as suitable numerical schemes for this kind of systems have been very interesting and challenging topics and have attracted the attention of many authors.

Recently, a robust numerical scheme was constructed for a model of two-phase flows, which is related to the model (1.1), see [40]. This scheme was shown to give a reasonable approximations

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and possesses many interesting properties: it preserves the positivity of the volume fractions in both phases, it can capture equilibrium states, preserves the positivity of the density, and satisfies the numerical minimum entropy principle in the first phase (the gas phase). However, numerical tests in [40] also showed that the scheme still provide approximate solutions which convergence to a function slightly different from the exact solution in some cases. In this work, we will build a scheme which can handle this problem: approximate solutions are convergent to the exact solution in all test cases observed. Unlike most existing schemes which treat the nonconservative terms via an intermediate computing step, or by intermediate states, a Godunov-type scheme can deal with the nonconservative terms by exact solutions of the local Riemann problems at grid nodes. Therefore, the difficulty lies on how to construct exact Riemann solutions of the model. Fortunately, this issues has recently been resolved in [36,39], where the Riemann problem for the model (1.1) was critically investigated. Motivated by these works, we will describe computational exact Riemann solutions, and then we build up a Godunov-type scheme relying on these computational exact solutions of local Riemann problems at grid nodes for (1.1) for initial data not only on the subsonic region, but also in the supersonic region. Then, we will prove that the scheme is *well-balanced* in the sense that it can capture *exactly* stationary contact waves in both phases. This means that

$$U_j^n = U_j^0,$$

for all integer j and for all $n = 1, 2, 3, \dots,$ where U_j^n is the approximation of the exact solution value $U(x_j, t^n)$. Furthermore, we provide numerical tests in both subsonic and supersonic regions. In each of these tests, we compute the errors for different mesh sizes. It is shown that the errors tend to zero as the mesh sizes go to zero in all the test. This indicates that the approximate solutions converge to the exact solution.

There have been many works in the literature on hyperbolic systems of balance laws in nonconservative form. Theoretical study for this kind of system was carried out in [17,22,25,26]. The Riemann problem for various hyperbolic systems of balance laws in nonconservative form were considered in [4,18,20,21,27,28,30,32,36,38,39]. Godunov-type schemes for various hyperbolic systems of balance laws in nonconservative form were studied in [2,14,29,33,34]. Various numerical schemes for two-phase flow models were considered in [1,3,9,10,13,15,16,19,31,35,37,40–42]. Well-balanced schemes for other nonconservative systems were studied in [5,6,11,12,23,24].

The organization of this paper is as follows. In Section 2 we will review basic properties of the system (1.1). Section 3 is devoted to the constructions of computational exact solutions of the Riemann problem for (1.1). In Section 4 we will construct a Godunov-type numerical scheme for (1.1). Numerical tests are presented in Section 5. Finally, we will present conclusions and discussions in Section 6.

2. Preliminaries

2.1. Characteristic fields

Throughout, we assume for simplicity that the fluid in each phase is isentropic and ideal, where the equation of state is

$$p_g = p_g(\rho_g) = \kappa_g {\rho_g}^{\gamma_g},\ \ p_s = p_s(\rho_s) = \kappa_s {\rho_s}^{\gamma_s},\ \ \kappa_g, \kappa_s > 0,\ \ \gamma_g, \gamma_s > 1.$$

The system (1.1) can be re-written as a system of balance laws in non-conservative form as

$$\partial_t \mathbb{U} + A(\mathbb{U})\partial_x \mathbb{U} = 0, \tag{2.1}$$

where $\mathbb{U} = (U_g, U_s, \alpha_g)^T$, $U_g = (\rho_g, u_g)$, $U_s = (\rho_s, u_s)$ and

$$A(\mathbb{U}) = \begin{pmatrix} u & \rho_g & 0 & 0 & \frac{\rho_g(u_g - u_s)}{\alpha_g} \\ \frac{p_g'(\rho_g)}{\rho_g} & u_g & 0 & 0 & 0 \\ 0 & 0 & u_s & \rho_s & 0 \\ 0 & 0 & \frac{p_s'(\rho_s)}{\rho_s} & u_s & \frac{p_g - p_s}{\alpha_s \rho_s} \\ 0 & 0 & 0 & 0 & u_s \end{pmatrix},$$

where $(.)'$ denotes the derivative of the function under consideration. The characteristic equation of the matrix $A(\mathbb{U})$ is given by

$$(u_s - \lambda)\Big((u_g - \lambda)^2 - p_g'(\rho_g)\Big)\Big((u_s - \lambda)^2 - p_s'(\rho_s)\Big) = 0.$$

Thus, we obtain five real eigenvalues

$$\begin{aligned} &\lambda_1(\mathbb{U}) = \lambda_1(U_g) = u_g - \sqrt{p_g'(\rho_g)}, \\ &\lambda_2(\mathbb{U}) = \lambda_2(U_g) = u_g + \sqrt{p_g'(\rho_g)}, \\ &\lambda_3(\mathbb{U}) = \lambda_3(U_s) = u_s - \sqrt{p_s'(\rho_s)}, \\ &\lambda_4(\mathbb{U}) = \lambda_4(U_s) = u_s + \sqrt{p_s'(\rho_s)}, \quad \lambda_5(\mathbb{U}) = \lambda_5(U_s) = u_s. \end{aligned} \tag{2.2}$$

The corresponding right eigenvectors can be chosen as

$$\begin{aligned} &r_1(\mathbb{U}) = \mu\Big(-\rho_g, \sqrt{p_g'(\rho_g)}, 0, 0, 0\Big)^T, \\ &r_2(\mathbb{U}) = \mu\Big(\rho_g, \sqrt{p_g'(\rho_g)}, 0, 0, 0\Big)^T, \\ &r_3(\mathbb{U}) = \nu\Big(0, 0, -\rho_s, \sqrt{p_s'(\rho_s)}, 0\Big)^T, \\ &r_4(\mathbb{U}) = \nu\Big(0, 0, \rho_s, \sqrt{p_s'(\rho_s)}, 0\Big)^T, \\ &r_5(\mathbb{U}) = \Big(-(u_g - u_s)^2\alpha_s\rho_g p_g, (u_g - u_s)\alpha_s p_g' p_s', (p_s - p_g)\alpha_g \\ &\qquad \times ((u_g - u_s)^2 - p_g'), 0, ((u_g - u_s)^2 - p_g')\alpha_g\alpha_s p_s'\Big)^T, \end{aligned} \tag{2.3}$$

where

$$\mu = \frac{2\sqrt{p_g'(\rho_g)}}{p_g''(\rho_g)\rho_g + 2p_g'(\rho_g)}, \quad \nu = \frac{2\sqrt{p_s'(\rho_s)}}{p_s''(\rho_s)\rho_s + 2p_s'(\rho_s)}.$$

It is not difficult to check that the eigenvectors $r_j(\mathbb{U})$, $j = 1, 2, 3, 4, 5$ are linearly independent. Thus, the system is hyperbolic. Furthermore, it holds that

$$\lambda_3(U_s) < \lambda_5(U_s) < \lambda_4(U_s).$$

It is interesting that $\lambda_5(U_s)$ may coincide with either $\lambda_1(u_g)$ or $\lambda_2(u_g)$ on a certain hyper-surface of the phase domain, called the *sonic surface* or *resonant surface*. We call the *supersonic region* to be the one in which

$$|u_g - u_s| > c := \sqrt{p_g'(\rho_g)}, \tag{2.4}$$

the *subsonic region* is the one in which $|u_g - u_s| < c$. To illustrate these regions, we consider the projection of the hyper-plane $u_s = u_{s*}$ of the phase domain, for an arbitrarily fixed u_{s*}, in the (ρ_g, u_g)−plane, see Fig. 1.

$$\begin{aligned} &G_1(u_{s*}) = \{(\rho_g, u_g): \quad u_g - u_{s*} > c\}, \\ &G_2(u_{s*}) = \{(\rho_g, u_g): \quad |u_g - u_{s*}| < c\}, \\ &G_3(u_{s*}) = \{(\rho_g, u_g): \quad u_g - u_{s*} < -c\}, \\ &\mathcal{C}_\pm(u_{s*}) = \{(\rho_g, u_g): \quad u_g - u_{s*} = \pm c\}. \end{aligned} \tag{2.5}$$

On the other hand, it is not difficult to verify that

$$\begin{aligned} &\nabla\lambda_j \cdot r_j = 1, \quad j = 1, 2, 3, 4, \\ &\nabla\lambda_5 \cdot r_5 = 0, \end{aligned} \tag{2.6}$$

so that the first, second, third, fourth characteristic fields (λ_j, r_j), $j = 1, 2, 3, 4$, are *genuinely non-linear*, while the fifth characteristic field (λ_5, r_5) is *linearly degenerate*.