Computers and Fluids 142 (2017) 72–78

Contents lists available at ScienceDirect

Computers and Fluids

journal homepage: www.elsevier.com/locate/compfluid

Finite element approximation of the viscoelastic flow problem: A non-residual based stabilized formulation

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ARTICLE INFO

Article history:
Received 21 October 2015
Revised 5 July 2016
Accepted 21 July 2016
Available online 22 July 2016

Keywords:
Viscoelastic fluids
Oldroyd-B fluids
Stabilized finite element methods
Variational multiscale

ABSTRACT

In this paper, a three-field finite element stabilized formulation for the incompressible viscoelastic fluid flow problem is tested numerically. Starting from a residual based formulation, a non-residual based one is designed, the benefits of which are highlighted in this work. Both formulations allow one to deal with the convective nature of the problem and to use equal interpolation for the problem unknowns $\boldsymbol{\sigma} - \boldsymbol{u} - p$ (deviatoric stress, velocity and pressure). Additionally, some results from the numerical analysis of the formulation are stated. Numerical examples are presented to show the robustness of the method, which include the classical 4: 1 planar contraction problem and the flow over a confined cylinder case, as well as a two-fluid formulation for the planar jet buckling problem.

1. Introduction

The finite element approximation of the flow of viscoelastic fluids presents several numerical difficulties, particularly for the spatial approximation. On the one hand, the finite element approximation used must satisfy two compatibility or inf-sup conditions, the first between velocity-pressure and the second associated to the interpolation of velocity and stress [1,2]. On the other hand, the convective nature presented both in the momentum and constitutive equation may lead to numerical instabilities.

The advective nature of the constitutive equation makes it necessary to use a stabilized formulation to avoid global oscillations. In the context of the finite element method, many algorithms have been developed to solve this problem: the classical SUPG method and its non-consistent counterpart, the SU method [3], the GLS method [4,5], and the stabilization based on the discontinuous Galerkin method [6]. In the present work, we apply two stabilized formulations based on the Variational Multiscale (VMS) framework introduced by Hughes et al. [7] for the scalar convection-diffusion-reaction problem, and extended later to the vectorial Stokes problem in [8], where the space of the sub-grid scales is taken orthogonal to the finite element space. As we shall see, this is an important ingredient in the design of our formulations.

The starting point of a VMS approach is to split the unknowns of the problem into two components, namely, the component that can be approximated by the finite element mesh and the unresolvable one, called sub-grid scale or simply sub-scale in what follows. The latter needs to be approximated in a simple manner in terms of the former, so as to capture its main effect and yield a stable formulation for the finite element unknown. There are different ways to approximate the sub-scale and, in particular, to choose the space where it is taken. We will describe two formulations which precisely differ in this choice. Both formulations will allow one to deal with the instabilities of the three-field viscoelastic formulation described earlier. There will be no need to meet the inf-sup conditions for the interpolation spaces and it will be possible to solve convection dominated problems both in the momentum and in the constitutive equation. For the latter, these methods have been found to work well. However, for the momentum equation we have observed that they are not robust in the presence of high gradients of the unknowns, and therefore we have had to modify them. The modification consists in designing a sort of term-by-term stabilization based on the choice of subscales orthogonal to the finite element space. We will describe in detail this method and the need for it.

The work is organized as follows. Section 2 contains the presentation of the problem. Section 3 presents our stabilized finite element approach. Section 4 is devoted to the numerical analysis results and Section 5 contains typical numerical examples used for this kind of fluids. Finally, in Section 6 conclusions are summarized.

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http://dx.doi.org/10.1016/j.compfluid.2016.07.012

2. The viscoelastic flow problem

2.1. Boundary value problem

To simulate the transitory incompressible and isothermal flow of viscoelastic fluids, one needs solve the momentum balance equation, the continuity equation and a constitutive equation that defines the viscoelastic contribution of the fluid.

The conservation equations for momentum and mass may be expressed as:

$$\rho\frac{\partial \boldsymbol{u}}{\partial t}+\rho\boldsymbol{u}\cdot\nabla\boldsymbol{u}-\nabla\cdot\boldsymbol{T}+\nabla p=\boldsymbol{f}\quad \text{in } \Omega,\ t\in]0,t_f[, \tag{1}$$

$$\nabla\cdot\boldsymbol{u}=0\quad \text{in } \Omega,\ t\in]0,t_f[\tag{2}$$

where Ω is the computational domain of $\mathbb{R}^d$ occupied by the fluid, $]0,\,t_f[$ is the time interval where the problem is solved, ρ denotes the constant density, $p:\Omega\to\mathbb{R}$ the pressure, $\boldsymbol{u}:\Omega\to\mathbb{R}^d$ the velocity vector, $\boldsymbol{f}:\Omega\to\mathbb{R}^d$ the force vector and $\boldsymbol{T}:\Omega\to\mathbb{R}^d\otimes\mathbb{R}^d$ the deviatoric extra stress tensor which can be defined in terms of a viscous and a viscoelastic or elastic contribution as:

$$\boldsymbol{T}=2\beta\eta_0\nabla^s\boldsymbol{u}+\boldsymbol{\sigma} \tag{3}$$

where $\beta\in[0,\,1]$ is a real parameter (ratio) to define the amount of viscous or solvent viscosity $\beta\eta_0$ and elastic or polymeric viscosity $(1-\beta)\eta_0$ in the fluid. For viscoelastic fluids, the problem is incomplete without the definition of a constitutive equation for the elastic part of the extra stress tensor ($\boldsymbol{\sigma}$). A large variety of approaches exist to define it (see [9,10] for a complete description). In this work, only the differential Oldroyd–B model is considered. It reads:

$$\begin{aligned}&\frac{\lambda}{2\eta_0}\frac{\partial\boldsymbol{\sigma}}{\partial t}+\frac{1}{2\eta_0}\boldsymbol{\sigma}-(1-\beta)\nabla^s\boldsymbol{u}\\&+\frac{\lambda}{2\eta_0}\left(\boldsymbol{u}\cdot\nabla\boldsymbol{\sigma}-\boldsymbol{\sigma}\cdot\nabla\boldsymbol{u}-(\nabla\boldsymbol{u})^T\cdot\boldsymbol{\sigma}\right)=\boldsymbol{0}\quad\text{in } \Omega,\ t\in]0,t_f[\end{aligned} \tag{4}$$

where λ is the relaxation time.

Calling $\boldsymbol{U}=[\boldsymbol{u},\boldsymbol{\sigma},p]$, $\boldsymbol{F}=[\boldsymbol{f},0,0]$ and defining

$$\mathcal{L}(\hat{\boldsymbol{u}},\boldsymbol{U}) := \begin{pmatrix}\rho\hat{\boldsymbol{u}}\cdot\nabla\boldsymbol{u}-2\beta\eta_0\nabla\cdot(\nabla^s\boldsymbol{u})-\nabla\cdot\boldsymbol{\sigma}+\nabla p\\ \nabla\cdot\boldsymbol{u}\\ \frac{\lambda}{2\eta_0}\frac{\partial\boldsymbol{\sigma}}{\partial t}+\frac{1}{2\eta_0}\boldsymbol{\sigma}-(1-\beta)\nabla^s\boldsymbol{u}+\frac{\lambda}{2\eta_0}\left(\hat{\boldsymbol{u}}\cdot\nabla\boldsymbol{\sigma}-\boldsymbol{\sigma}\cdot\nabla\hat{\boldsymbol{u}}-\left(\nabla\hat{\boldsymbol{u}}\right)^T\cdot\boldsymbol{\sigma}\right)\end{pmatrix} \tag{5}$$

and

$$T(\boldsymbol{U}) := \begin{pmatrix}\rho\frac{\partial\boldsymbol{u}}{\partial t}\\ 0\\ \frac{\lambda}{2\eta_0}\frac{\partial\boldsymbol{\sigma}}{\partial t}\end{pmatrix} \tag{6}$$

we may write (1), (2) and (4) using the definition (3) as $T(\boldsymbol{U})+\mathcal{L}(\hat{\boldsymbol{u}},\boldsymbol{U})=\boldsymbol{F}$.

Initial and boundary conditions have to be appended to problem (1)–(4). For simplicity in the exposition, we will consider the simplest boundary condition $\boldsymbol{u}=\boldsymbol{0}$ on $\partial\Omega$, and no boundary conditions for $\boldsymbol{\sigma}$. However, due to the hyperbolic nature of the constitutive equation, in some examples the elastic stresses are fixed on the inflow part of the boundary. For a complete description of the mathematical structure of the problem we refer to [11].

2.2. The variational form

We introduce some notation in order to write the weak form of the problem. The space of square integrable functions in a domain ω is denoted by $L^2(\omega)$, and the space of functions whose distributional derivatives of order up to $m\geq 0$ (integer) belong to $L^2(\omega)$ by $H^m(\omega)$. The space $H_0^m(\omega)$ consists of functions in $H^1(\omega)$ vanishing on $\partial\omega$. The topological dual of $H_0^1(\Omega)$ is denoted by $H^{-1}(\Omega)$, the duality pairing by $\langle\,\cdot,\,\cdot\rangle$, and the L^2 inner product in ω (for scalar, vectors and tensors) is denoted by $(\,\cdot,\,\cdot)$. The norm in a functional space X will be denoted by $\|\cdot\|_X$, except when $X=L^2(\Omega)$, case in which the subscript will be omitted.

Using this notation, the stress, velocity and pressure functional spaces for the continuous problem are $\boldsymbol{\Upsilon}_0=L^2\left(0,t_f;\,(H_0^1(\Omega))^{d\times d}_{\text{sym}}\right)$ (the subscript sym standing for symmetric tensors), $\boldsymbol{\mathcal{V}}_0=L^2\left(0,t_f;\,(H_0^1(\Omega))^d\right)$ and $\mathcal{Q}=\mathcal{D}\left(0,t_f;\,(L^2(\Omega)/\mathbb{R})\right)$ ($\mathcal{D}$ standing for distributions), respectively. The weak form of the problem consists in finding $\boldsymbol{U}=[\boldsymbol{u},p,\boldsymbol{\sigma}]\in\boldsymbol{\mathcal{X}}=\boldsymbol{\mathcal{V}}_0\times\mathcal{Q}\times\boldsymbol{\Upsilon}_0$, such that:

$$\begin{aligned}&\left(\rho\frac{\partial\boldsymbol{u}}{\partial t},\boldsymbol{v}\right)+2(\beta\eta_0\nabla^s\boldsymbol{u},\nabla^s\boldsymbol{v})+\langle\rho\boldsymbol{u}\cdot\nabla\boldsymbol{u},\boldsymbol{v}\rangle\\&+(\boldsymbol{\sigma},\nabla^s\boldsymbol{v})-(p,\nabla\cdot\boldsymbol{v})=\langle\boldsymbol{f},\boldsymbol{v}\rangle\end{aligned} \tag{7}$$

$$(q,\nabla\cdot\boldsymbol{u})=0 \tag{8}$$

$$\begin{aligned}&\left(\frac{\lambda}{2\eta_0}\frac{\partial\boldsymbol{\sigma}}{\partial t},\boldsymbol{\tau}\right)+\left(\frac{1}{2\eta_0}\boldsymbol{\sigma},\boldsymbol{\tau}\right)-((1-\beta)\nabla^s\boldsymbol{u},\boldsymbol{\tau})\\&+\frac{\lambda}{2\eta_0}\left(\boldsymbol{u}\cdot\nabla\boldsymbol{\sigma}-\boldsymbol{\sigma}\cdot\nabla\boldsymbol{u}-(\nabla\boldsymbol{u})^T\cdot\boldsymbol{\sigma},\boldsymbol{\tau}\right)=0\end{aligned} \tag{9}$$

for all $\boldsymbol{V}=[\boldsymbol{v},q,\boldsymbol{\tau}]\in\hat{\boldsymbol{\mathcal{X}}}$ (the time independent counterpart of $\boldsymbol{\mathcal{X}}$), where $\boldsymbol{f}\in\left(0,t_f;\,(H^{-1}(\Omega))\right)$.

In a compact form, the problem (7)–(9) can be written as:

$$\left(\rho\frac{\partial\boldsymbol{u}}{\partial t},\boldsymbol{v}\right)+\left(\frac{\lambda}{2\eta_0}\frac{\partial\boldsymbol{\sigma}}{\partial t},\boldsymbol{\tau}\right)+B(\boldsymbol{u};\boldsymbol{U},\boldsymbol{V})=\langle\boldsymbol{f},\boldsymbol{v}\rangle$$

for all $\boldsymbol{V}\in\hat{\boldsymbol{\mathcal{X}}}$, where

$$\begin{aligned}B\left(\hat{\boldsymbol{u}};\boldsymbol{U},\boldsymbol{V}\right)=&\,2(\beta\eta_0\nabla^s\boldsymbol{u},\nabla^s\boldsymbol{v})+\left\langle\rho\hat{\boldsymbol{u}}\cdot\nabla\boldsymbol{u},\boldsymbol{v}\right\rangle+(\boldsymbol{\sigma},\nabla^s\boldsymbol{v})\\&-(p,\nabla\cdot\boldsymbol{v})+(q,\nabla\cdot\boldsymbol{u})\\&+\left(\frac{1}{2\eta_0}\boldsymbol{\sigma},\boldsymbol{\tau}\right)-((1-\beta)\nabla^s\boldsymbol{u},\boldsymbol{\tau})\\&+\frac{\lambda}{2\eta_0}\left(\hat{\boldsymbol{u}}\cdot\nabla\boldsymbol{\sigma}-\boldsymbol{\sigma}\cdot\nabla\hat{\boldsymbol{u}}-\left(\nabla\hat{\boldsymbol{u}}\right)^T\cdot\boldsymbol{\sigma},\boldsymbol{\tau}\right)\end{aligned} \tag{10}$$

3. Numerical approximation

3.1. Galerkin finite element discretization

The standard Galerkin approximation for the variational problem can be performed by considering a finite element partition $\mathcal{T}_h$ of the domain Ω. The diameter of an element domain $K\in\mathcal{T}_h$ is denoted by h_K and the diameter of the element partition is defined by $h=\max\{h_K\mid K\in\mathcal{T}_h\}$. Under the above considerations, we can construct conforming finite elements spaces, $\boldsymbol{\mathcal{V}}_{h,0}\subset\boldsymbol{\mathcal{V}}_0$, $\mathcal{Q}_h\subset\mathcal{Q}$ and $\boldsymbol{\Upsilon}_{h,0}\subset\boldsymbol{\Upsilon}_0$ in the usual manner. If $\boldsymbol{\mathcal{X}}_h=\boldsymbol{\mathcal{V}}_{h,0}\times\mathcal{Q}_h\times\boldsymbol{\Upsilon}_{h,0}$, and $\boldsymbol{U}_h=[\boldsymbol{u}_h,p_h,\boldsymbol{\sigma}_h]$, the Galerkin finite element approximation consist in finding $\boldsymbol{U}_h\in\boldsymbol{\mathcal{X}}_h$, such that:

$$\left(\rho\frac{\partial\boldsymbol{u}_h}{\partial t},\boldsymbol{v}_h\right)+\left(\frac{\lambda}{2\eta_0}\frac{\partial\boldsymbol{\sigma}_h}{\partial t},\boldsymbol{\tau}_h\right)+B(\boldsymbol{u}_h;\boldsymbol{U}_h,\boldsymbol{V}_h)=\langle\boldsymbol{f},\boldsymbol{v}_h\rangle \tag{11}$$

for all $\boldsymbol{V}_h=[\boldsymbol{v}_h,q_h,\boldsymbol{\tau}_h]\in\hat{\boldsymbol{\mathcal{X}}}_h$.

In principle, we have posed no restrictions on the choice of the finite element spaces. However, there are restrictions that must be satisfied explicitly in the discrete formulation used. These are the same as for the three-field formulation of the Stokes problem