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On the consistency and convergence of particle-based meshfree discretization schemes for the Laplace operator

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ABSTRACT

The Laplace operator appears in the governing equations of continua describes dissipative dynamics, and it also emerges in some second order partial differential equations such as the Poisson equation. In this paper, accuracy and its convergence rates of some meshfree discretization schemes for the Laplace operator are studied as a verification. Moreover, a novel meshfree discretization scheme for the second order differential operator which enables us to use smaller dilation parameter of the compact support of the weight function is proposed, and its application for the meshfree discretization of the Poisson equation demonstrates an improvement of the solution accuracy.

1. Introduction

The Laplace operator, or the Laplacian, denoted by symbols ∇^2 or Δ, is a second order differential operator given by the divergence of the gradient of a function in d-dimensional Euclidian space. It appears in a lot of second order differential equations that describe physical phenomena, and they can be categorized into three types: parabolic, hyperbolic, and elliptic type. These are important mathematical representations of physics. Furthermore, in the field of fluid dynamics, the Laplace operator occurs in the viscosity (dissipative dynamics) and the pressure Poisson equation; therefore, numerical analyses of differential equations with the Laplace operator, and their discretization procedures, are of interest for computational fluid dynamics.

Various numerical methods have been developed for the solution of ordinary/partial differential equations. For instance, the Finite Element Method(FEM), the Finite Difference Method(FDM), and the Finite Volume Method(FVM), are widely utilized methodologies. Their common feature is that they divide a spatial domain into a set of discrete subdivisions so-called mesh, grid, or cell, which requires pre-defined and fixed connectivity of nodes. Alternatively, variety of meshfree methods, which establish a system of algebraic equations without the use of pre-defined mesh/grid/cell, have been vigorously sought in order to find better discretization procedures without mesh-constraints. Particle methods based on the Lagrangian description are one of the meshfree methods which make the most of meshfree talent, and their most important advantage is that they can easily handle simulations of very large deformations, even with the changes of the topological structure and fragmentation-coalescence of continua.

The Smoothed Particle Hydrodynamics(SPH) method [1,2] and the Moving Particle Semi-implicit(MPS) method [3,4] are extensively used strong-form meshfree and particle methods for numerical analysis of fluid flow with free surfaces. Although they have been shown to be useful in engineering applications, their standard formulae of spatial discretization schemes of the gradient, the divergence, and the Laplace operators lack polynomial completeness (reproducing conditions); therefore, inconsistencies of spatial discretization procedures have been resulted in adverse effects for both computational accuracy and stability. In order to overcome the inconsistency problem of meshfree discretizations, the (weighted) least squares method or equivalents have been utilized in various meshfree and/or particle methods (e.g. [5–14]), and Least Squares Moving Particle Semi-implicit (LSMPS) method [15] is one of the methods based on the least squares technique. Enhancement of accuracy and stability compared with the existing MPS method [3,4] were demonstrated through a applicative validation test problem [15]; however, convergence tests for the second order derivative approximations of the LSMPS method have not been well examined.

In this paper, the consistency of the second order derivative approximations for the strong-form particle methods including the

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LSMPS method are focused on, and the convergence of some meshfree particle discretization schemes for the Laplace operator are studied as a verification. Moreover, a novel meshfree discretization scheme for the second order differential operator which enables us to use smaller dilation parameter of the compact support of the weight function is proposed.

2. Meshfree discretization schemes for the Laplace operator

In this section, an overview of meshfree spatial discretization schemes for the Laplace operator (second order derivative approximations of the SPH method, the MPS method, and the LSMPS method) is presented. Furthermore, a novel scheme for the second order differential operator which enables us to utilize smaller dilation parameter of the compact support of the weight function is derived.

2.1. The Smoothed Particle Hydrodynamics method approximations

The Smoothed Particle Hydrodynamics (SPH) method was originally developed for the field of astrophysics by Lucy [1], and Gingold and Monaghan [2] in 1977, and it has been applied for numerical analyses of fluid flows and structures (Early contributions have been reviewed in several articles, for instance, [16–20]). In what follows, the SPH method is presented as one of the meshfree discretization methods.

The SPH interpolation is based on the following simple concept,

$$f(\mathbf{x}) = \int_{\mathbb{R}^d} \delta(\mathbf{x}' - \mathbf{x}) f(\mathbf{x}') d\mathbf{x}', \quad \forall \mathbf{x} \in \mathbb{R}^d, \tag{1}$$

where $\delta(\mathbf{x})$ is the Dirac delta distribution, and the key ingredient of the SPH method is to replace the Dirac delta distribution with well-behaved *smoothing* kernel function $w(\mathbf{x};\, h)$ (h is a smoothing length) that mimics the useful properties of the Dirac delta distribution. Whereafter, a SPH smoothing interpolation is discretized for a set of scattered nodes $\{\mathbf{x}_j\}_{1 \le j \le N}$, i.e.,

$$\langle f(\mathbf{x}) \rangle = \int_{\mathbb{R}^d} w(\mathbf{x}' - \mathbf{x};\, h) f(\mathbf{x}') d\mathbf{x}', \tag{2}$$

$$\approx \sum_j w(\mathbf{x}_j - \mathbf{x};\, h) f_j \Delta V_j, \tag{3}$$

where ΔV_j denotes nodal measures of point $\mathbf{x}_j$, and $f_j = f(\mathbf{x}_j)$. Note that a part $w(\mathbf{x}_j - \mathbf{x};\, h)\Delta V_j$ in Eq. (3) can be seen as a shape function in the finite element discretization procedure. The derivatives of a function are obtained by differentiating the discrete interpolated function, for instance, the gradient of a function f is approximated by

$$\langle \nabla f(\mathbf{x}) \rangle \approx \sum_j \left[\nabla w(\mathbf{x}_j - \mathbf{x};\, h) \right] f_j \Delta V_j. \tag{4}$$

Following the same procedure yields a SPH approximation for the Laplacian of a function f:

$$\left\langle \nabla^2 f(\mathbf{x}) \right\rangle \approx \sum_j \left[\nabla^2 w(\mathbf{x}_j - \mathbf{x};\, h) \right] f_j \Delta V_j. \tag{5}$$

This formulation, however, is not used today since its accuracy and solution stability depend strongly on the nodal distribution [16,21,22] and it lacks polynomial completeness conditions. Alternatively, some approximating discretization schemes for the Laplace operator have been utilized in the SPH method.

2.1.1. Brookshaw type SPH formula

Brookshaw [23] proposed the following SPH formulation for the approximation of the Laplacian of a function:

$$\left\langle \nabla^2 f(\mathbf{x}) \right\rangle_i \approx \sum_{j \in \Lambda_i} \frac{\nabla_i w(\mathbf{x}_j - \mathbf{x}_i;\, h) \cdot (\mathbf{x}_j - \mathbf{x}_i)}{\|\mathbf{x}_j - \mathbf{x}_i\|^2} \left(f_j - f_i \right) \Delta V_j, \tag{6}$$

$$= \sum_{j \in \Lambda_i} \left[\frac{1}{r_{ij}} \frac{\partial w(\mathbf{x}_j - \mathbf{x}_i;\, h)}{\partial r_{ij}} \Delta V_j \right] (f_j - f_i), \tag{7}$$

where $\nabla_i = \nabla|_{\mathbf{x}=\mathbf{x}_i}$, $r_{ij} = \|\mathbf{x}_j - \mathbf{x}_i\|$, and Λ_i stands for sets of neighboring nodes $\mathbf{x}_j$ that locate in the compact support of the kernel function of the node $\mathbf{x}_i$. It should be mentioned that this discretization scheme forms a finite difference like formula:

$$\left\langle \nabla^2 f(\mathbf{x}) \right\rangle_i \approx \sum_{j \in \Lambda_i} C_{ij} (f_j - f_i), \tag{8}$$

and various discretization schemes based on finite difference like formulae have been proposed for the SPH method (e.g.[24,25]). Morris[26] applied this brookshaw type SPH formula of the approximating Laplace operator for discretization of the viscosity term in the Navier–Stokes equations, and this scheme [23,26] is one of the most widely used formulae in the SPH method for discretization of the Laplacian; therefore, the accuracy of Brookshaw type SPH Laplace operator (Eq. (6)) will be investigated numerically, later in this study.

2.1.2. Monaghan and Gingold type SPH formula

Monaghan and Gingold [27] proposed the following formula for discretization of the viscosity term of fluid flows:

$$\left\langle \mu \nabla^2 \mathbf{u} \right\rangle_i \approx 2(d+2)\mu \sum_{j \in \Lambda_i} \nabla_i w(\mathbf{x}_j - \mathbf{x}_i;\, h) \frac{(\mathbf{x}_j - \mathbf{x}_i) \cdot (\mathbf{u}_j - \mathbf{u}_i)}{\|\mathbf{x}_j - \mathbf{x}_i\|^2} \Delta V_j, \tag{9}$$

$$= 2(d+2)\mu \sum_{j \in \Lambda_i} \left[\frac{\nabla_i w(\mathbf{x}_j - \mathbf{x}_i;\, h) (\mathbf{x}_j - \mathbf{x}_i)^T}{\|\mathbf{x}_j - \mathbf{x}_i\|^2} \right] (\mathbf{u}_j - \mathbf{u}_i) \Delta V_j, \tag{10}$$

$$= 2(d+2)\mu \sum_{j \in \Lambda_i} \left[\frac{1}{r_{ij}^3} \frac{\partial w(\mathbf{x}_j - \mathbf{x}_i;\, h)}{\partial r_{ij}} \Delta V_j (\mathbf{x}_j - \mathbf{x}_i)(\mathbf{x}_j - \mathbf{x}_i)^T \right] (\mathbf{u}_j - \mathbf{u}_i), \tag{11}$$

where μ denotes the dynamic viscosity, and $\mathbf{u}$ stands for the velocity vector. This formulation for discretization of the viscosity term is also a widely utilized scheme in the SPH method, however, this model do not satisfy the Stokes hypothesis as pointed out by Colagrossi et al. [28]. The consequence of this fact have not been studied deeply and its study of accuracy and inconsistency in the Stokes hypothesis will be postponed for future studies.

2.2. The Moving Particle Semi-implicit method approximations

The Moving Particle Semi-implicit method [3,4] was developed by Koshizuka and Oka for numerical analyses of incompressible flows with free surfaces. Since the MPS method requires to discretize the pressure Poisson equation and the viscosity term, variety of spatial discretization schemes for the Laplace operator have been proposed.

2.2.1. Koshizuka and Oka type formula

Koshizuka and Oka [3] proposed the following “particle interaction model” to discretize the Laplace operator:

$$\left\langle \nabla^2 f(\mathbf{x}) \right\rangle_i \approx \frac{2d}{\lambda n} \sum_{j \in \Lambda_i} w(\mathbf{x}_j - \mathbf{x}_i;\, h)(f_j - f_i), \tag{12}$$