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An improved accurate monotonicity-preserving scheme for the Euler equations

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ABSTRACT

The accurate monotonicity-preserving (MP) scheme of Suresh and Huynh (1997) [5] is a high-order and high-resolution method for hyperbolic conservation laws. However, the robustness of the MP scheme is not very high. In this paper, a detailed analysis on this scheme is performed, and two potential causes which may account for the weak robustness are revealed. Furthermore, in order to enhance the robustness of the MP scheme, an improved version of the MP scheme is presented, in which a strict continuous total-variation-diminishing (TVD) numerical flux is used at a disturbed discontinuity so that oscillations cannot grow indefinitely without violating the TVD condition. Without destroying the very high resolution property, numerical tests show that the improved scheme shares a strong robustness in simulating extreme numerical tests.

1. Introduction

As is well known, flows involving multi-scales and discontinuities occur widely in many natural phenomena and engineering applications [1,2]. In terms of simulating the flows, numerical schemes must be high-order accurate in smooth regions to resolve the multi-scales, and essentially oscillation-free near discontinuities to capture features such as shock waves. Consequently, the development of high-order shock-capturing schemes is of importance.

Van Leer [3] first showed that it is beneficial to strive for schemes with a high order of accuracy while capturing shock waves in an essentially oscillation-free manner. By using piecewise linear interpolation, coupled with a limiting strategy to control oscillations at discontinuities, he designed a second-order monotonicity-preserving version of the Godunov scheme. Later, Colella and Woodward [4] developed the piecewise parabolic method (PPM), which employs a four-point centered stencil to define the numerical flux; this numerical flux is then limited to control oscillations. Following this limiting approach, Suresh and Huynh [5] proposed the accurate monotonicity-preserving (MP) scheme. In the MP scheme, starting with a primary numerical flux calculated by any high-order scheme, the following two procedures are executed: (1) calculate a local interval, which is designed by enlarging the first-order monotonicity-preserving interval derived in [5], and (2) maintain/replace the primary numerical flux according to the relation between the primary numerical flux and this local interval: (i) if the primary numerical flux lies in this local interval, it is maintained, and (ii) if the primary numerical flux is beyond this local interval, it is replaced by the nearest bound of this local interval. The key feature of the MP scheme is that this local interval is designed to have the following property—it is very large in smooth regions so that it contains the primary numerical flux, and automatically degenerates to the monotonicity-preserving interval at discontinuities. That is to say, this local interval will enlarge for non-monotonic data in order to achieve the accuracy-preserving property, and automatically degenerates to the monotonicity-preserving interval for monotonic data. Due to this property, the total variation of the numerical solution in the MP scheme is allowed to increase only for non-monotonic data, and diminish for monotonic data [6].

The recent development of the MP scheme mainly focuses on the problem of calculating the primary numerical flux. In the original paper, the fifth-order upwind scheme is adopted [5]. Later, some other schemes are tested, such as the fifth-order compact upwind scheme [7] or center schemes with controllable

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artificial dissipation [8,9]. Balsara and Shu [10] even adopt a high-order weighted essentially non-oscillatory (WENO) scheme to obtain the primary numerical flux, yielding high-order monotonicity-preserving WENO schemes. Daru and Tenaud [6] reinterpreted the local interval in MP schemes as TVD-like conditions, and applied these conditions to a one-step scheme.

The MP scheme shows high-resolution for multi-scale problems. Therefore, it is widely adopted in practical applications [11,12]. However, the robustness of the MP scheme is not very high, compared with the popular WENO methods [13]. In this paper, we perform a detailed analysis of the MP scheme, trying to reveal the potential causes which may account for the weak robustness of the MP scheme. Furthermore, in order to enhance the robustness of the MP scheme, a simple modification is proposed to suppress the numerical oscillations more efficiently and/or to prevent the appearance of numerical oscillations as possible. Numerical tests show that the improved MP scheme shows a more strong robustness.

2. Method

2.1. Framework

Consider the following one-dimensional linear advection equation

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 \tag{1}$$

with constant advection speed, i.e. $\frac{df}{du} = a$. Without loss of generality, assume that $a \geq 0$. Eq. (1) is discretized in a uniform grid defined by the points $x_i = i\Delta x$, $i = 1, \cdots, N$, with cell boundaries given by $x_{i+1/2} = x_i + \frac{\Delta x}{2}$, where Δx is the uniform grid spacing. The spatial discretization is obtained by implicitly defining the numerical flux function $h(x)$ as

$$f(x) = \frac{1}{\Delta x}\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} h(\xi)\mathrm{d}\xi, \tag{2}$$

such that the spatial derivative in Eq. (1) is exactly approximated by a conservative finite difference formula at the cell boundaries,

$$\frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = -\frac{h_{i+1/2} - h_{i-1/2}}{\Delta x}, \tag{3}$$

where $h_{i\pm\frac{1}{2}} = h(x_{i\pm\frac{1}{2}})$, and $u_i(t)$ is a numerical approximation to the point value $u(x_i, t)$. In practice, Eq. (3) is approximated as

$$\frac{\mathrm{d}u_i(t)}{\mathrm{d}t} \approx -\frac{\widehat{f}_{i+1/2} - \widehat{f}_{i-1/2}}{\Delta x}, \tag{4}$$

where numerical fluxes $\widehat{f}_{i\pm\frac{1}{2}}$, reconstructed from known cell average values f_i, are approximations of $h_{i\pm\frac{1}{2}}$. Then Eq. (4) can be marched by one time step or substep of TVD Runge–Kutta schemes [14]:

$$u_i^{n+1} \approx u_i^n - \frac{\Delta t}{\Delta x}\left(\widehat{f}^n_{i+1/2} - \widehat{f}^n_{i-1/2}\right), \tag{5}$$

where Δt is the time step, u_i^{n+1} the numerical approximation to the point value $u(x_i, t)$ at time level t^{n+1}, and $\widehat{f}^n_{i\pm\frac{1}{2}}$ is the numerical flux at time level t^n (the superscript n in $\widehat{f}^n_{i\pm\frac{1}{2}}$ is omitted for brevity hereafter).

2.2. The accurate monotonicity-preserving (MP) scheme

When the MP scheme [5] is used to calculate the numerical flux $\widehat{f}_{i+1/2}$ in Eq. (5), it is implemented as follows.

2.2.1. Some notations used in the MP scheme

First of all, we introduce some definitions used in the accurate monotonicity-preserving (MP) scheme. The first one is the minmod function for q arguments [15]:

$$\mathrm{minmod}(z_1, z_2, \cdots z_q) := s \cdot \min(|z_1|, |z_2|, \cdots, |z_q|), \tag{6}$$

where

$$s = \frac{1}{2}(sgn(z_1) + sgn(z_2))\left|\frac{1}{2}(sgn(z_1) + sgn(z_3))\cdots\frac{1}{2}(sgn(z_1) + sgn(z_q))\right|,$$

and $sgn(z)$ is a function that returns the sign of the argument z. The second one is the definition of an interval [15]:

$$I[z_1, z_2, \cdots, z_q] := [\min(z_1, z_2, \cdots, z_q), \max(z_1, z_2, \cdots, z_q)]. \tag{7}$$

The third one is the definition of the local curvature $d^{MM}_{i+1/2}$ [5]:

$$d^{MM}_{i+1/2} := \mathrm{minmod}(d_i, d_{i+1}) \tag{8}$$

where $d_i = f_{i+1} + f_{i-1} - 2f_i$.

2.2.2. Review of the MP scheme

First, obtain the primary numerical flux $\widehat{f}^o_{i+1/2}$ using a high-order scheme. In the original paper of the MP scheme [5], the following fifth-order upwind scheme (U5)

$$\widehat{f}^o_{i+1/2} = \frac{2f_{i-2} - 13f_{i-1} + 47f_i + 27f_{i+1} - 3f_{i+2}}{60} \tag{9}$$

is used. Then, the primary numerical flux $\widehat{f}^o_{i+1/2}$ is maintained or replaced according to the following limiting procedures.

First, in order to achieve the monotonicity-preserving property: (I) $\widehat{f}^o_{i+1/2}$ should lie between f_i and f_{i+1}. (II) u_i^{n+1} should lie between u_{i-1} and u_i, which merely ensures that $\widehat{f}^o_{i+1/2}$ lies between f_i and f^{UL}, where

$$f^{UL} = f_i + \kappa(f_i - f_{i-1}), \tag{10}$$

and $\kappa \geq 2$ [5]. Combining the two assumptions, a first-order monotonicity-preserving interval $I[f_i, f^{MP}]$, which is just the intersection of $I[f_i, f_{i+1}]$ and $I[f_i, f^{UL}]$, is derived, where

$$f^{MP} = f_i + \mathrm{minmod}(f_{i+1} - f_i, \kappa(f_i - f_{i-1})). \tag{11}$$

However, to bring $\widehat{f}^o_{i+1/2}$ into the interval $I[f_i, f^{MP}]$ will result in degeneration to first order near an extremum. In order to avoid this drawback, Suresh and Huynh [5] proposed the idea of enlarging the intervals defined above to avoid the loss of accuracy. Based on the parabolic interpolation, $I[f_i, f_{i+1}]$ and $I[f_i, f^{UL}]$ were enlarged to $I[f_i, f_{i+1}, f^{MD}]$ and $I[f_i, f^{UL}, f^{LC}]$, respectively, where

$$f^{MD} = \frac{1}{2}(f_i + f_{i+1}) - \frac{1}{2}d^{MM}_{i+1/2}, \tag{12}$$

$$f^{LC} = f_i + \frac{1}{2}(f_i - f_{i-1}) + \frac{4}{3}d^{MM}_{i-1/2}, \tag{13}$$

or [6]

$$f^{LC} = \frac{1}{2}(f_i + f^{UL}) + \frac{\kappa}{2}d^{MM}_{i-1/2}. \tag{13'}$$

Moreover, it has been proven that $I[f_i, f_{i+1}, f^{MD}]$ and $I[f_i, f^{UL}, f^{LC}]$ will enlarge only for non-monotonic numerical data, and will automatically degenerate to $I[f_i, f_{i+1}]$ and $I[f_i, f^{UL}]$ for monotonic numerical data [5,6]. However, in practice it is recommended to replace $d^{MM}_{i+1/2}$ with a more restrictive but heuristic measure of the local curvature $d^{M4}_{i+1/2}$ [5]

$$d^{M4}_{i+1/2} = \mathrm{minmod}(d_i, d_{i+1}, 4d_i - d_{i+1}, 4d_{i+1} - d_i). \tag{14}$$