Computers and Fluids 140 (2016) 81–96

Contents lists available at ScienceDirect

Computers and Fluids

journal homepage: www.elsevier.com/locate/compfluid

An efficient adaptive high-order scheme based on the WENO process

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ARTICLE INFO

Article history:
Received 20 December 2015
Revised 15 June 2016
Accepted 10 September 2016
Available online 12 September 2016

Keywords:
Adptive
WENO
High order
Computational fluid dynamics

ABSTRACT

In this paper, we propose an efficient adaptive scheme based on the WENO (weighted essentially non-oscillatory) process for the hyperbolic conservation laws. This scheme achieves fifth order accuracy as the fifth order WENO scheme does. Also, due to its adaptive mechanism, the scheme deals with problems which contain both discontinuities and complex smooth regions much better than traditional high order schemes. In addition, the proposed scheme saves computational costs by avoiding solving redundant nonlinear weights in certain places. Systematic analyses and numerical tests show that our adaptive scheme is of a high level of robustness and resolution.

1. Introduction

Nowadays, the hyperbolic conservation laws in the form:

$$u_t + \nabla \cdot f(u) = 0 \tag{1}$$

arise in many practical applications such as magneto hydrodynamics (MHD), shallow water problems, haptotaxis models, traffic models, and computational aero-acoustics(CAA), etc. There are no analytical solutions to these complicated problems. Therefore, the demand for the high-resolution methods has grown rapidly in the last decades. Among these high resolution methods, the high order schemes are more welcome because they can provide the same result with less grid points for the steady problems and cause less damping for the unsteady ones.

For high order schemes, the most straightforward approach to increase the order of accuracy is to extend the computational stencil for the finite volume/difference methods [1,2]. Although this approach is efficient and easy to work, it is inclined to obtain spurious oscillations when it is applied to problems with strong discontinuities. Also, it is a shortcoming that the solver attempts to build a high order reconstruction using data from regions of the flow that may not be physically relevant [1].

To avoid the unphysical relevance and make the computational stencil compact, some other methods, such as the Discontinuous Galerkin (DG) method [3], the Spectral Difference (SD) method [4], and the Spectral Volume (SV) method [5], have been studied widely. But all these methods suffer two primary drawbacks: a high storage requirement and a lack of robustness.

On the other hand, the TVD concept [6] was proposed to be free from such spurious oscillations. However, it falls to the first order accuracy at extrema. The ENO(essentially non-oscillatory) scheme and the TVB concept [7] were introduced to overcome this defect. Also, the WENO-class schemes were proposed by using a convex combination of all candidate stencils instead of choosing the smoothest one to improve the levels of accuracy and robustness near discontinuities.

The classical WENO scheme, introduced by Jiang and Shu [8], can obtain $(2r-1)th$ order accuracy. Henrick et al. [9] developed a mapping function to refine the accuracy and the convergence order at critical points. Borges et al. [10] devised a different weighting formulation to satisfy the same condition as Henric‘s. However, because the TVB concept allows oscillations when the spurious oscillations do not grow unboundedly, all these WENO-class schemes are inevitable to encounter undershoot or overshoot phenomenon which influence the robustness and the accuracy intensely. Furthermore, all the WENO-class schemes are based on the local characteristic decomposition, which often causes excessive numerical dissipation. Moreover, the calculation costs for this procedure and the nonlinear weights are very high. To reduce the excessive numerical dissipation and the large computational cost, researchers have developed a class of hybrid schemes by using the WENO scheme to capture the discontinuities and adopting the other high order schemes in smooth regions [11–17]. For example, Matthew R. Norman [18] developed a WENO-limited method within the finite volume (FV) framework for parallel strategy. In addition, Robert Nichols [19] and his coworkers interpolated the primitive variables

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http://dx.doi.org/10.1016/j.compfluid.2016.09.011

to the grid half nodes by adopting the WENO reconstruction, which is more efficient by avoiding the expensive projections onto the characteristic states. However, most aforementioned schemes' performances depend on the artificial parameters strongly. Inappropriate parameters' definitions may overlook the oscillation phenomenon and lower the level of robustness. To properly deal with these issues, we propose an efficient adaptive scheme in this paper. To be with a high level of robustness and efficiency, this scheme identifies where the discontinuity is and where the spurious oscillation phenomenon appears automatically. Also, it avoids adopting the expensive characteristic transformation and the problem-dependent parameters.

The present paper is organized as follows: After a brief review of WENO class methods for one-dimensional scalar conservation laws in Section 2, the high order TVD reconstructions is presented concisely in Section 3. In Section 4, the time evolution method is briefly shown. The effective adaptive high-order scheme is derived in section 5 where a detailed adaptation algorithm and a convergence study are introduced. In Section 6, some numerical results are presented to verify the characteristics of the new scheme. And the conclusions are drawn in Section 7.

2. Weighted essentially non-oscillatory schemes

In this section, we briefly describe the fifth-order WENO scheme for one dimensional scalar conservation laws. The one dimensional hyperbolic conservation law (1), with a uniform grid size Δx, has the semi-discretization form

$$\frac{du_i(t)}{dt} = -\frac{1}{\Delta x}\left(\widehat{f}_{i+\frac{1}{2}}\left(u^+_{i+\frac{1}{2}}, u^-_{i+\frac{1}{2}}\right) - \widehat{f}_{i-\frac{1}{2}}\left(u^+_{i-\frac{1}{2}}, u^-_{i-\frac{1}{2}}\right)\right) \tag{2}$$

where $\widehat{f}_{i+\frac{1}{2}}$ describes the numerical flux of u between the cells i and $i+1$.

For a general flux, we can split it into two parts either globally or locally:

$$\widehat{f}(u^+, u^-) = f(u^+) + f(u^-), \tag{3}$$

where $\frac{df(u^+)}{du} \geq 0$ and $\frac{df(u^-)}{du} \leq 0$. As can be seen in the Ref [1], the difference between different monotone fluxes for the high order reconstruction methods makes little difference. Thus, we choose the commonly used global Lax–Friedrich flux splitting method in this paper

$$f(u^\pm) = \frac{1}{2}(f(u) \pm \alpha u) \tag{4}$$

where $\alpha = \max|f'(u)|$ and the maximum is taken over the whole relevant range of u. Here we only describe how u^+ is computed and drop the $''+''$ sign in the superscript for simplicity.

To satisfy the conservative property of the spatial discretization, we implicitly define a cell averaged function $h(x)$ as

$$u(x) = \frac{1}{\Delta x}\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} h(\xi)d\xi, \tag{5}$$

thus

$$u(x)_x = \frac{1}{\Delta x}\left(h\left(x+\frac{\Delta x}{2}\right) - h\left(x-\frac{\Delta x}{2}\right)\right). \tag{6}$$

The fifth-order polynomial approximation is built through the convex combination of interpolated values $\widehat{u}^k(x_{i\pm1/2})(k=0,1,2)$, in which the third order interpolation polynomial on each stencil is $S_k^3 = (x_{i+k-2}, x_{i+k-1}, x_{i+k})$, and

$$\widehat{u}_{i\pm\frac{1}{2}} = \sum_{k=0}^{2}\omega_k \widehat{u}^k\left(x_{i\pm\frac{1}{2}}\right), \tag{7}$$

where the nonlinear weights ω_k satisfy

$$\sum_{k=0}^{2}\omega_k = 1.\ \omega_k \geq 0, \quad k = 0,1,2 \tag{8}$$

The polynomial $\widehat{u}^k(x)$ at the cell boundary $x_{i+\frac{1}{2}}$ in sub-stencil S_k can be constructed as

$$\widehat{u}^k\left(x_{i+\frac{1}{2}}\right) = \widehat{u}^k_{i+\frac{1}{2}} = \sum_{j=0}^{2} c_{kj}u_{i-k+j}, \quad i = 0, ...N, \tag{9}$$

where the c_{kj} are Lagrange interpolation coefficient depending on the left-shift parameter k. The specific form can be written as

$$\begin{aligned}
\widehat{u}^0_{i+\frac{1}{2}} &= \frac{1}{6}(2u_{i-2} - 7u_{i-1} + 11u_i)\\
\widehat{u}^1_{i+\frac{1}{2}} &= \frac{1}{6}(-u_{i-1} + 5u_i + 2u_{i+1})\\
\widehat{u}^2_{i+\frac{1}{2}} &= \frac{1}{6}(2u_i + 5u_{i+1} - u_{i+2}),
\end{aligned} \tag{10}$$

which can be shown in the following form by Taylor series expansion:

$$\begin{aligned}
\widehat{u}^0_{i\pm\frac{1}{2}} &= u_{i\pm\frac{1}{2}} - \frac{1}{4}u'''(0)\Delta x^3 + O(\Delta x^4)\\
\widehat{u}^1_{i\pm\frac{1}{2}} &= u_{i\pm\frac{1}{2}} + \frac{1}{12}u'''(0)\Delta x^3 + O(\Delta x^4)\\
\widehat{u}^2_{i\pm\frac{1}{2}} &= u_{i\pm\frac{1}{2}} - \frac{1}{12}u'''(0)\Delta x^3 + O(\Delta x^4)
\end{aligned} \tag{11}$$

The nonlinear weights are defined as

$$\omega_k = \frac{\alpha_k}{\sum_{l=0}^{2}\alpha_l}, \quad \omega_k = \frac{d_k}{(\beta_k + \varepsilon)^p} \tag{12}$$

where the ideal weights, $d_0 = \frac{3}{10}, d_1 = \frac{6}{10}, d_2 = \frac{1}{10}$, are the coefficients which can generate the central upstream fifth-order scheme for the 5-points stencil S_5. The parameter ε is defined as 10^{-6} to prevent the denominator from becoming zero in the WENO-JS scheme (the classical WENO proposed by Jiang and Shu) and 10^{-40} in the WENO-Z scheme.

2.1. The WENO(JS) scheme

The smoothness indicators β_k (k = 1,2,3) proposed by Jiang and Shu are given by

$$\beta_k = \sum_{l=1}^{2}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \Delta x^{2l-1}\left(\frac{d^l}{dx^l}\widehat{u}^k\right)^2 dx \tag{13}$$

To take (13) in an explicit form,

$$\begin{aligned}
\beta_0 &= \frac{1}{4}(u_{i-2} - 4u_{i-2} + 3u_i)^2 + \frac{13}{12}(u_{i-2} - 2u_{i-2} + u_i)^2\\
\beta_1 &= \frac{1}{4}(u_{i-1} - u_{i+1})^2 + \frac{13}{12}(u_{i-1} - 2u_i + u_{i+1})^2\\
\beta_2 &= \frac{1}{4}(3u_i - 4u_{i+1} + 3u_{i+2})^2 + \frac{13}{12}(u_i - 2u_{i+1} + u_{i+2})^2
\end{aligned} \tag{14}$$

it is obvious that β_k satisfies $\beta_k = D(1 + O(\Delta x^2))$ where D is some non-zero quantity independent of k. As is shown by Henrick et al. [9], the WENO weights ω_k satisfy the condition $\omega_k = d_k + O(\Delta x^2)$, which provide sufficient conditions

$$\begin{aligned}
&\sum_{k=0}^{2}A_k(\omega^+ - \omega^-) = O(\Delta x^3)\\
&\omega^\pm - d_k = O(\Delta x^2)
\end{aligned} \tag{15}$$

for the fifth convergence order in the smooth region. However, the convergence order is degraded at critical points where the derivative vanishes. On the other hand, if a discontinuity occurs in one