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Application of positivity-preserving well-balanced discontinuous Galerkin method in computational hydrology

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ABSTRACT

The positivity-preserving well-balanced discontinuous Galerkin (DG) method (Xing et al. J Sci Comput **57**, 2013) is employed to solve the shallow water equations on an unstructured triangular mesh and to study their applications in computational hydrology. The grid convergence of the DG method is verified via the steady state oblique hydraulic jump problem. The dam-breaking problems with wet and dry river beds are conducted to demonstrate the positivity-preserving property of the scheme. The tidal bores in an idealized estuary problem are simulated to study the development and evolution of the tidal bores from different amplitudes of incoming tidal waves and topography of the river bed bottom. The numerical experiments above demonstrate that the DG method can be applied successfully to these class of problems in computational hydrology.

1. Introduction

In the field of hydrology, there is a class of special flows in which the water level and velocity change rapidly and abruptly. In these flows, strong gradients or discontinuities often appear in the flows, for example, a sharp water level drop due to the discontinuities in the initial condition resulting from a breaking of a dam, and a development of sharp tidal bores due to the strong nonlinearity interaction between an outgoing river water and an incoming ocean tidal waves at the estuary. The tidal bores are a tidal phenomenon along a coast where a river empties into a sea in which the leading edge of the incoming high tide forms a wave (or waves) of water that travels up the river against the direction of the river current (also known as positive surge). However, there are only a few places, under some specific geological conditions, where the tidal bores can occur—not all coast features tidal bores. The river must be fairly shallow and has a narrow outlet to the sea. The estuary, the area where the river meets the sea, must be wide and flat. The coast's tidal wave length, the distance between high and low tides must be quite large (typically at least 6*m*). When all of these conditions are satisfied, a tidal bore is formed. A few notable tidal bore systems in the world are the Pororoca Tidal Bore with Amazon river, Brazil, Bono Tidal Bore with Kampar River, Indonesia, Severn Tidal Bore with Severn River, United Kingdom, Silver Dragon Tidal Bore with Qiantang River, China and Turnagain Arm Tidal Bore with Cook Inlet, Alaska. An exception of the conditions above is the Amazon River where the mouth of the river is not narrow but shallow and dotted by many low-lying islands and sand bars. We refer the reader to the national geographic website[1] for more descriptions about the development and evolution of the tidal bores in the nature.

The research of the tidal bores has two basic goals. Academical-wise, the study of strong intermittent flow such as the tidal bores has been an important and challenging subject in hydrology. Research on the numerical simulations of tidal bores, not only can reveal the water dynamics of nonlinear wave laws, but also has an academic significance and a theoretical value in hydrodynamics and computational fluid dynamics. Engineering-wise, the study of the formation and evolution of the tidal bores, the in-depth understanding of its effect on estuarine environment and the effect of human activities on the tidal bores, can enhance the human survival and the economic and social development related to water environmental problems, preventing the potential harmful effects from the tidal bores while keeping this unique and valuable natural resource.

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[1] http://education.nationalgeographic.com/encyclopedia/tidal-bore/

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The shallow water equations serve as a very important model in the simulations of flows in the rivers, lakes and coastal areas, including the tidal bores. Although the shallow water equations have been studied extensively in the past two decades, they remain an active area of research in both theoretical studies and numerical simulations due to their practical importance. The traditional numerical methods for solving the shallow water equations are the method of characteristics (MOC), finite element method (FEM) and finite different Method (FDM). These methods have been successful in the simulations of many continuous flows, but are not satisfactory in solving discontinuous flows like the tidal bores. On one hand, one difficulty encountered in the simulation of shallow water equations is how to exactly balance the flux gradients by the source terms in the steady-state solution. The well-balanced schemes [1–6] are specially designed to preserve exactly the steady-state solution up to the machine error with relatively coarse meshes. Moreover, dry areas (where the water height is exactly equal to zero) might appear in the natural environments such as the dam-breaking problem over a dry river bed. Due to the Gibbs phenomenon when using a high order scheme without employing some forms of limiting on the solution and/or flux (limiter), a non-physical negative water height will be generated numerically in the simulations. It causes problem in calculating the eigenvalues that involve a square root of the water height. Therefore, many positivity-preserving schemes [7–11] were designed to preserve the positivity of certain physical quantities, such as the mass fraction in a reactive Euler equations and the water height in the shallow water equations with a dry area. A few of existing numerical methods [12–18] are able to maintain both the well-balance and positivity of the numerical schemes simultaneously.

Discontinuous Galerkin (DG) method is a class of finite element methods using discontinuous piecewise polynomial space as the solution and test function spaces (see [19] for a historic review and basic idea). It has been used extensively in solving the shallow water equations [20–26]. Recently, the positivity-preserving well-balanced DG method for the shallow water equations [16] was proposed to maintain the still water steady state solution exactly, and to preserve the non-negativity of the water height without a loss of mass conservation. In [17], a simple positivity-preserving limiter was extended to the DG method on the unstructured triangular meshes to guarantee the positivity of the water height.

In this study, we investigate the application of the positivity-preserving well-balanced DG method designed in [17] to the computational hydrology on the unstructured grids. We employ this method for simulating several challenging practical engineering problems such as the dam-breaking problems with wet and dry river beds and the development and evolution of the tidal bores in an idealized estuary. The grid convergence of the DG method is verified in the case of steady state oblique hydraulic jump. The positivity-preserving limiter is used in the DG method to avoid the non-physical negative water height in the simulations of dam-breaking problems with a dry river bed. The evolution of the flooding of the wet and dry river beds shows different flow structures that are unique in each individual case. A tidal bore is simulated to study the formation and evolution of the tidal bores with a trumpet-like shape river mouth emptying into an ocean while subjected to a large incoming tidal wave, similar to the one at the Qiantang River, China. We study factors like the tidal amplitude and river topography, which are related to the propagation and breaking of the undular tidal bores while moving up to the river against the current.

The paper is organized as follows. In Section 2, a very brief introduction to the positivity-preserving well-balanced DG method for the shallow water equations will be given. In Section 3, a classical two-dimensional steady state example is presented to validate the accuracy and convergence of the DG method. The positivity-preserving well-balanced DG method is then applied to the dam-breaking problems with both wet and dry river beds. Also, an idealized estuary problem that simulates the formation and evolution of the tidal bores in a long straight river is shown. Finally, conclusion and future work are given in Section 4.

2. Positivity-preserving well-balanced DG method

The two-dimensional shallow water equations take the form

$$\begin{cases} h_t + (hu)_x + (hv)_y = 0 \\ (hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x + (huv)_y = -ghb_x \\ (hv)_t + (huv)_x + \left(hv^2 + \frac{1}{2}gh^2\right)_y = -ghb_y, \end{cases} \tag{1}$$

where h is the water height, $(u, v)^T$ is the velocity vector, $b(x, y)$ is the bottom topography and g is the gravitational constant. In a compact form, (1) can be written as

$$Q_t + \nabla \cdot \mathbf{F}(Q) = \mathbf{S}(h, b),$$

where $Q = (h, hu, hv)^T$ with the superscript T denoting the transpose, $\mathbf{F}(Q) = (f(Q), g(Q))$ is the flux vector and $\mathbf{S}(h, b)$ is the source term.

Let $\mathcal{T}_\tau$ be a family of triangular partitions of the computational domain Ω parameterized by $\tau > 0$. For any triangle $K \in \mathcal{T}_\tau$, we define $\tau_K := \text{diam}(K)$ and $\tau := \max_{K\in\mathcal{T}_\tau} \tau_K$. For each edge e_K^i $(i = 1, 2, 3)$ of K, we denote its length by l_K^i, and outward unit normal vector by $\mathbf{n}_K^i$. Let $K(i)$ be the neighboring triangle along the edge e_K^i and $|K|$ be the area of the triangle K. In a high order DG method, we seek an approximation, still denoted by Q with an abuse of notation, which belongs to the finite dimensional space:

$$V_\tau = \{w \in L^2(\Omega);\ w|_K \in P^k(K)\ \ \forall K \in \mathcal{T}_\tau\}, \tag{2}$$

where $P^k(K)$ denotes the space of polynomials of degree at most k on K.

Let $\mathbf{x}$ denotes (x, y), the standard DG scheme is given by

$$\iint_K Q_t w\, d\mathbf{x} - \iint_K \mathbf{F}(Q)\cdot\nabla w\, d\mathbf{x} + \sum_{i=1}^{3}\int_{e_K^i} \widehat{\mathbf{F}}|_{e_K^i}\cdot \mathbf{n}_K^i w\, ds = \iint_K \mathbf{S} w\, d\mathbf{x}, \tag{3}$$

where $w(\mathbf{x})$ is a test function, and the numerical flux $\widehat{\mathbf{F}}$ is defined by

$$\widehat{\mathbf{F}}|_{e_K^i}\cdot\mathbf{n}_K^i = \mathcal{F}(Q_i^{int(K)}, Q_i^{ext(K)}, \mathbf{n}_K^i), \tag{4}$$

where $Q_i^{int(K)}$ and $Q_i^{ext(K)}$ are the approximations to the values on the edge e_K^i obtained from the interior and the exterior of K. We could, for example, use the simple global Lax–Friedrichs flux

$$\mathcal{F}(a_1, a_2, \mathbf{n}) = \frac{1}{2}[\mathbf{F}(a_1)\cdot\mathbf{n} + \mathbf{F}(a_2)\cdot\mathbf{n} - \alpha(a_2 - a_1)],$$

where $\alpha = \max\left((|u| + \sqrt{gh}, |v| + \sqrt{gh})\cdot\mathbf{n}\right)$ and the maximum is taken over the whole region. Notice that $h \geq 0$ should be a non-negative value at all time.

In order to achieve the well-balanced property, we are interested in preserving the still water stationary solution, namely,

$$h + b = const, \qquad u = v = 0, \tag{5}$$

exactly. Well-balanced numerical methods are designed in [16,17], and take the form

$$\iint_K Q_t w\, d\mathbf{x} - \iint_K \mathbf{F}(Q)\cdot\nabla w\, d\mathbf{x} + \sum_{i=1}^{3}\int_{e_K^i} \widehat{\mathbf{F}}^*|_{e_K^i}\cdot \mathbf{n}_K^i w\, ds = \iint_K \mathbf{S} w\, d\mathbf{x}, \tag{6}$$

where the well-balanced numerical fluxes $\widehat{\mathbf{F}}^*$ are given by

$$\widehat{\mathbf{F}}^*|_{e_K^i}\cdot\mathbf{n}_K^i = \mathcal{F}(Q_i^{*,int(K)}, Q_i^{*,ext(K)}, \mathbf{n}_K^i) + \langle\delta_{i,x}^*, \delta_{i,y}^*\rangle\cdot\mathbf{n}_K^i, \tag{7}$$