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A post-processing technique for stabilizing the discontinuous pressure projection operator in marginally-resolved incompressible inviscid flow

Sumedh M. Joshi [a,*], Peter J. Diamessis [b], Derek T. Steinmoeller [c], Marek Stastna [c], Greg N. Thomsen [d]

[a] Center for Applied Mathematics, 657 Rhodes Hall, Cornell University, Ithaca NY, 14850, United States
[b] School of Civil and Environmental Engineering, 105 Hollister Hall, Cornell University, Ithaca NY, 14850, United States
[c] Department of Applied Mathematics, University of Waterloo, Ontario, CA, Canada
[d] Applied Research Laboratories, The University of Texas at Austin, Austin, TX, United States

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A method for post-processing the velocity after a pressure projection is developed that helps to maintain stability in an under-resolved, inviscid, discontinuous element-based simulation for use in environmental fluid mechanics process studies. The post-processing method is needed because of spurious divergence growth at element interfaces due to the discontinuous nature of the discretization used. This spurious divergence eventually leads to a numerical instability. Previous work has shown that a discontinuous element-local projection onto the space of divergence-free basis functions is capable of stabilizing the projection method, but the discontinuity inherent in this technique may lead to instability in under-resolved simulations. By enforcing inter-element discontinuity and requiring a divergence-free result in the weak sense only, a new post-processing technique is developed that simultaneously improves smoothness and reduces divergence in the pressure-projected velocity field at the same time. When compared against a non-post-processed velocity field, the post-processed velocity field remains stable far longer and exhibits better smoothness and conservation properties.

1. Introduction

Pressure-projection methods are a class of numerical techniques that decouple the solution of pressure from velocity in the numerical simulation of an incompressible flow. These methods, first developed by Chorin [3], overcome the difficult problem of the pressure-velocity coupling through the incompressibility constraint, and are widely used in computational fluid dynamics for simulating time-dependent flow [11,12]. The fundamental idea in the pressure-projection method is that the momentum equation and the incompressibility constraint are time-integrated separately and in sequence. First, the momentum equations are solved to advance the velocity to an interstitial time. Next, a projection operation maps these velocities onto the space of divergence-free functions by way of solving a Poisson equation and advances the velocity to the next time. Thus, projection methods are also sometimes known as fractional-step or time-splitting methods [16,17,20] and are widely-used solving viscous, time-dependent, incompressible flows. Because of their ubiquity, much work has been to done to construct consistent boundary conditions for the pressure and velocity [2,16,17,19,24,26] and stable spatial discretizations of each constituent operator in the time-splitting [5]. These efforts are largely focused on avoiding the spurious divergence boundary layers that can form when inconsistent boundary conditions are used in the time-splitting within the projection method [16,26]. A concise review of pressure projection methods can be found in Ref. [11].

While in theory the velocity, once projected, is supposed to be divergence-free, it has recently been observed that in the discontinuous Galerkin (DG) formulation the projection operation may contain non-solenoidal eigenmodes [25]. Thus, the projected velocity fields themselves are not exactly divergence-free, which can lead to numerical instability and inaccuracy . A possible explanation for the cause (which is discussed in Section 2.2) of this is that this is due to the discontinuous spatial discretization of the Laplacian operator within the pressure projection method. To remedy this, Steinmoeller et al. [25] construct a post-processing method

* Corresponding author. Tel.: +7135945390.
E-mail address: smj96@cornell.edu (S.M. Joshi).

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that explicitly projects the velocity field onto a computed basis set that is exactly the null-space basis of the discrete DG divergence operator, and show that this projection eliminates the spurious divergence due to the non-solenoidal eigenmodes of the pressure projection update operator. While this post-processing technique is effective in eliminating non-solenoidal components of the velocity, by virtue of the discontinuous nature of the spatial discretization, the exact null-space projection (ENP) as in Steinmoeller et al. does not take into account continuity between elements. ENP is a projection that is entirely local to an element.

In this work, we ask whether this locality is ever problematic, and if so what should be done to address it. While this question is broad in scope, it is shown that at least in one instance of a marginally resolved simulation the discontinuity in the ENP can lead to instability. As a remedy, a modified null-space projection technique is used as a post-processing method that explicitly takes into account inter-element continuity in a regularized least-squares sense. It is shown that in this instance the so-called weak null-space projection method appears to yield greater stability as a post-processing technique. In this regard, this line of reasoning is in parallel to previous efforts which focused on constructing boundary conditions to avoid the spurious divergence boundary layers [16,26], but focusing instead on the spurious divergence that is observed to form at inter-element boundaries.

To study the relative merits of the exact null-space projection and its modification that captures inter-element continuity, the 2D incompressible Euler equations are used as a proxy for the full incompressible Navier–Stokes equations. The incompressible Euler equations model an inviscid incompressible flow with a stratified background density profile that is not dependent on time and only depends on the vertical direction; a perturbation density, $\rho'(\mathbf{x}, \mathbf{t})$, is overlaid on the background stratification and as noted does vary in time and space. The equations are given as

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{u} \cdot \nabla \mathbf{u} - \frac{\mathbf{1}}{\rho_\mathbf{0}} \nabla \mathbf{p} - \mathbf{g} \frac{\rho'}{\rho_\mathbf{0}} \mathbf{e_z} \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

$$\frac{\partial \rho'}{\partial t} = -\nabla \cdot \mathbf{u}(\rho' + \overline{\rho}) \tag{3}$$

where $\rho(x, z, t) = \rho_0 + \overline{\rho}(z) + \rho'(x, z, t)$ is density stratified in the Boussinesq approximation [18] with $\overline{\rho}(z)$ the background stratification, $\mathbf{u}(\mathbf{x}, \mathbf{z}, \mathbf{t})$ the velocity, p the pressure, and $\mathbf{e_z}$ the unit vector in the vertical direction. It is assumed that $\rho_0 \gg \overline{\rho} \gg \rho'$. In the pressure-projection method Eqs. (1) and (2) are not solved directly. Instead, a projection operator $\mathbb{P}$ is used to solve

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbb{P}\left(\mathbf{u} \cdot \nabla \mathbf{u} - \mathbf{g} \frac{\rho'}{\rho_\mathbf{0}} \mathbf{e_z} \right) \tag{4}$$

in which the projection operator $\mathbb{P}$ is defined as

$$\mathbb{P}\mathbf{u} := \mathbf{u} - \nabla \mathbf{\Delta}^{-\mathbf{1}} (\nabla \cdot \mathbf{u}). \tag{5}$$

This decouples the solution of pressure from velocity, and allows for a sequential solution algorithm in which the velocity is advected by the nonlinear term prior to being projected into a divergence-free space by $\mathbb{P}$.

While viscosity is neglected here, the difficulties encountered in stabilizing the pressure projection method for the Navier–Stokes equations are all encountered here as well. The presence of viscosity will only aid in damping the numerical instabilities driven by the nonlinear advection term, so stability in the inviscid case is more difficult to achieve due to the absence of physical viscous dissipation. In fact, the discussion of stability and under-resolution is primarily manifest in advection-dominated Navier–Stokes simulations in which a broad range of scales is present due to the lack of strong viscosity; in this sense then, the incompressible Euler simulations presented here capture the essence of the difficulty in simulating incompressible fully-viscous flows.

The numerical method used to model the density-stratified inviscid incompressible Euler equations is the spectral multi-domain penalty method (SMPM), a high-order discontinuous collocation-based variant of the spectral element method [10,13] which has been previously shown to be effective in simulating high-Reynolds number environmental flows [7,8] using the pressure-projection method. In particular, we will use as an example the inviscid propagation of an internal solitary wave in a density-stratified channel as a test bed for evaluating the efficacy of the various post-processing methods. In these simulations, the initial conditions propagate as waves in a non-dispersive non-dissipative fashion through the domain while retaining their form. Thus, the degree to which these solutions maintain their structure is a good heuristic for the efficacy of these post-processing methods. It should be mentioned that although the SMPM is chosen to demonstrate these methods, these ideas are readily extensible to discontinuous Galerkin (DG) discretizations, and especially to high-order DG discretizations.

The paper is organized as follows. In Section 2 the exact and weak null-space projection methods, their motivation and the notation used are described. In Section 3 two simulations with each of the three methods are conducted and compared. Both simulations are of the same propagating internal solitary wave in tank, and the simulations differ in their mesh resolution. In Section 4 contains a discussion of the results presented in Section 3, along with a computational assessment of the spectrum and numerical conditioning properties of all of the methods compared in this paper. Finally, we conclude with a short discussion of applicability to other numerical methods as well as a discussion of future work related to the ideas outlined herein.

2. Methods

This section summarizes the exact null-space projection (ENP) as outlined in Ref. [25], the weak null-space projection which is the contribution of this work, and the numerical method.

2.1. Numerical method and notation

In the 2D SMPM, each element is assumed to be smoothly and invertibly mapped from the unit square $[-1, 1] \times [-1, 1]$ and the element connectivity is logically cartesian (each element has a single neighbor in each of the North, South, East, and West directions). Within each element lies a two-dimensional Gauss–Lobatto–Legendre (GLL) grid; denote as n the number of GLL points per direction per element, and m_x and m_z the number of x and z elements in the grid[1]. Thus, the total number of grid points is $r = n^2 m_x m_z$. On the GLL grid, a two-dimensional nodal Lagrange interpolant basis of polynomial order $n + 1$ is constructed such that each basis function has unit value on one of the n^2 GLL points and zero on all of the others. This nodal basis is used for approximating functions and their derivatives which are calculated by way of spectral differentiation matrices [4] which compute derivatives of nodally-represented functions by multiplying the nodal values by derivatives of the Lagrange interpolants themselves. The SMPM is a discontinuous method and so C^0/C^1 inter-element continuity and boundary conditions are only weakly enforced, which yields

[1] Here z is the vertical direction as is convention in environmental fluid mechanics.