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A high-order Discontinuous Galerkin solver for unsteady incompressible turbulent flows

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ABSTRACT

In this work we investigate the use of adaptive linearly implicit Rosenbrock-type Runge–Kutta and Explicit Singly Diagonally Implicit Runge–Kutta schemes to integrate in time high-order Discontinuous Galerkin space discretizations of the incompressible Navier–Stokes (INS) and Reynolds Averaged Navier–Stokes (URANS) equations. The objective of this activity is to assess the efficiency and accuracy of the considered schemes coupled with a time-step adaptation technique for incompressible URANS simulations. The schemes have been first investigated for the computation of the laminar travelling waves and of the turbulent flow around a circular cylinder at a Reynolds number $Re = 5 \times 10^4$, verifying the convergence order, a simple relation to set the system tolerance starting from the tolerance of the adaptation strategy, and their computational efficiency. Finally, the best scheme resulting from our analysis has been applied to the URANS simulation of the flow through a vertical axis wind turbine, comparing the results with CFD and experimental data available in literature.

1. Introduction

Discontinuous Galerkin (DG) methods, due to the ever increasing available computational power and the need for better accuracy, emerged as one of the most promising approaches to high-fidelity fluid dynamic computations in many technical areas, such as aeronautics, aeroacoustics and turbomachinery [1–7].

These computations require robust and accurate long time integration of unsteady flows, characterized by a wide range of temporal scales. In this context some features of the DG methods, *e.g.* compactness and flexibility, can be advantageous both for explicit and implicit time integration approaches. Explicit schemes, even if simple to implement and of high accuracy, can be limited by time-step restrictions, while the implicit schemes, even if memory consuming due to the need of the Jacobian matrix, can represent a viable solution.

Large scale unsteady computations can be efficiently performed by means of projection methods [8], see *e.g.* [9,10] for formulations employing spatial discontinuous Galerkin discretizations. Nevertheless, since in the context of decoupled time integration strategies the achievement of high-order accurate time integration is more tricky (for instance the boundary condition treatment is not trivial and might harm pressure accuracy), we adopt a monolithic velocity–pressure space couple [11]. This choice allows to exploit all the implicit time integration schemes available in literature but requires to solve the momentum equation coupled with the incompressibility constraint. Clearly, without an effective preconditioning strategy, this approach cannot guarantee the computational efficiency of the projection methods.

The most popular implicit scheme is the Backward Differentiation Formulae (BDF) [12], which are only *A*-stable up to the second-order, and their low accuracy is not well suited for the increasing required level of resolution. Several high-order implicit time integrators, relying on multistage and multistep schemes, are already available and their coupling with the DG space discretization has been analysed: Explicit Singly Diagonally Implicit Runge–Kutta (ESDIRK) schemes (*A*-stable up to order five) [13], Modified Extended BDF (MEBDF) schemes (*A*-stable up to order four) [14,15], and Two Implicit Advanced Step-point (TIAS) schemes (*A*-stable up to order six) [16,17]. At each time-step, all these schemes require to solve several nonlinear systems of equations, a task that can be efficiently performed, for example, by means of the (quasi-)Newton method. In the large family of implicit Runge–Kutta methods, the class of linearly implicit Rosenbrock-type Runge–Kutta schemes, [13,18], is receiving increasing attention because such methods, being linearly implicit, require to solve only linear systems in the

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stages within each time-step, *i.e.* the Jacobian matrix needs to be assembled and factored only once per time step.

All these scheme have been mainly investigated for the solution of the compressible Navier–Stokes equations, without techniques to adapt the time-step. A few information can be found in literature regarding their behaviour in the solution of Differential Algebraic Equations (DAE) and stiff systems arising from the discretization of the unsteady Reynolds Average Navier–Stokes (URANS). To cover this gap, this paper focus on the implementation and assessment of Runge–Kutta schemes (ESDIRK and Rosenbrock) as time integrators for the high-order DG space discretization of the incompressible URANS equations closed by the k–$\widetilde{\omega}$ turbulence model [19]. The robustness and efficiency of the time integration schemes have been enhanced adopting a technique to adapt the time-step, based on a local error estimator which exploits the local truncation error of the time integration scheme and of its lower order embedded scheme. In this work we have carried out a comparative assessment, in terms of accuracy and performance, of three schemes: the fourth order/six stages (ESDIRK) [20], the third order/three stages (ROS3PL) [21] and fourth order/six stages (RODASP) [22] linearly implicit one-step Rosenbrock methods. These scheme are suitable for DAE systems arising from the discretization of the incompressible Navier–Stokes (INS) equations, they have an embedded scheme for the time-step adaptation strategy and the ROS3PL and RODASP schemes can handle also time-dependent boundary conditions without or with slight order reduction, respectively.

The reliability, robustness and accuracy of the proposed implementation have been assessed by computing several incompressible unsteady test cases: (i) the laminar travelling waves on a doubly-periodic unit square; (ii) the turbulent flow around a circular cylinder for a Reynolds number $Re = 5 \times 10^4$; and (iii) the turbulent flow through a vertical axis wind turbine (VAWT). The travelling waves problem was used to investigate (i) the temporal order of convergence of the schemes, (ii) a relation between the tolerance of the time-step adaptation technique and the tolerance used to solve the system arising from the temporal discretization, and (iii) the computational efficiency of the schemes to obtain a given accuracy with the time-step adaptation technique. The circular cylinder was used to investigate the computational efficiency of the schemes with time-step adaptation technique for the computation of turbulent flows. Finally, the turbulent flow through a VAWT was computed with the best performing scheme, comparing results with numerical and experimental data available in literature.

The rest of the paper is organized as follows. Section 2 describes the DG space discretization method for the Incompressible Navier–Stokes, RANS and turbulence model equations, while Section 3 illustrates the high-order time integration schemes implemented and the adaptive time-step algorithm. Numerical results are shown in Section 4, and Section 5 contains the conclusions.

2. DG space discretization

Incompressible Navier–Stokes, RANS and turbulence model equations can be written in compact form as

$$\frac{\partial \widehat{\mathbf{q}}}{\partial t} + \nabla \cdot \mathbf{F}_c(\mathbf{q}) + \nabla \cdot \mathbf{F}_v(\mathbf{q}, \nabla \mathbf{q}) + \mathbf{s}(\mathbf{q}, \nabla \mathbf{q}) = \mathbf{0}, \tag{1}$$

where $\mathbf{q} \in \mathbb{R}^m$ denotes the vector of the m primitive variables $(p, u_i, k, \omega)^T$ (for $i = 1, d$), $\mathbf{s} \in \mathbb{R}^m$ the source term, d the space dimension. $\mathbf{F}_c, \mathbf{F}_v \in \mathbb{R}^M \otimes \mathbb{R}^N$ denote the inviscid and viscous flux functions, while $\widehat{\mathbf{q}} \in \mathbb{R}^m$ is defined as $(0, u_i, k, \omega)^T$.

To discretize the governing equations in space the system (1) is firstly multiplied by an arbitrary smooth test function $\mathbf{v} = \{v_1, \ldots, v_m\}$ and then integrated by parts, thus obtaining its weak form. The solution $\mathbf{q}$ and the test function $\mathbf{v}$ are then replaced with a finite element approximation $\mathbf{q}_h$ and a discrete test function $\mathbf{v}_h$ both belonging to $\mathbf{V}_h \overset{\text{def}}{=} [\mathbb{P}_d^k(\mathcal{T}_h)]^m$, where

$$\mathbb{P}_d^k(\mathcal{T}_h) \overset{\text{def}}{=} \left\{ v_h \in L^2(\Omega) \,|\, v_{h|K} \in \mathbb{P}_d^k(K),\ \forall K \in \mathcal{T}_h \right\} \tag{2}$$

is the discrete polynomial space in physical coordinates. $\mathbb{P}_d^k(K)$ denotes the restriction of the polynomial functions of d variables and total degree $\leq k$ to the element K belonging to the triangulation $\mathcal{T}_h = \{K\}$, consisting of a set of non-overlapping arbitrarily shaped and possibly curved elements, built on an approximation Ω_h of the domain Ω. We also define as $\mathcal{F}_h$ the set of the mesh faces $\mathcal{F}_h \overset{\text{def}}{=} \mathcal{F}_h^i \cup \mathcal{F}_h^b$, where $\mathcal{F}_h^b$ collects the faces located on the boundary of Ω_h and for any $F \in \mathcal{F}_h^i$ there exist two elements $K^+, K^- \in \mathcal{T}_h$ such that $F \in \partial K^+ \cap \partial K^-$. Moreover, for all $F \in \mathcal{F}_h^b$, $\mathbf{n}_F$ is the unit outward normal to Ω_h, whereas, for all $F \in \mathcal{F}_h^i$, $\mathbf{n}_F^-$ and $\mathbf{n}_F^+$ are the unit outward normals pointing to K^+ and K^-, respectively. To deal with discontinuous functions over the internal faces $F \in \mathcal{F}_h^i$ we introduce the jump $[\![\cdot]\!]$ and average $\{\cdot\}$ trace operators, that is

$$[\![v_h]\!] \overset{\text{def}}{=} v_{h|K^+}\mathbf{n}_F^+ + v_{h|K^-}\mathbf{n}_F^-, \qquad \{v_h\} \overset{\text{def}}{=} \frac{v_{h|K^+} + v_{h|K^-}}{2}. \tag{3}$$

When applied to vector functions these operators act componentwise.

Following the approach presented in [23], for each equation of the system, and without loss of generality, we choose the set of test and shape functions in any element K coincident with the set $\{\phi\}$ of N_{dof}^K orthogonal and hierarchical basis functions in that element. Such basis is built by means of the modified Gram–Schmidt (MGS) algorithm (Ref. [24]) starting from a set of monomials defined over each elementary space $\mathbb{P}_d^k(K)$ in a reference frame relocated in the element barycentre and aligned with the principal axes of inertia of K.

Each component $q_{h,j}$, $j = 1, \ldots, m$, of the numerical solution $\mathbf{q}_h \in \mathbf{V}_h$ can be expressed, in terms of the elements of the global vector $\mathbf{Q}$ of unknown degrees of freedom, as $q_{h,j} = \phi_l Q_{j,l}$, $l = 1, \ldots, N_{dof}^K$, $\forall K \in \mathcal{T}_h$.

Accounting for these aspects, the DG discretization of the RANS and turbulence model equations consists in seeking, for $j = 1, \ldots, m$, the elements of $\mathbf{Q}$ such that

$$\begin{aligned}
&\sum_{K \in \mathcal{T}_h} \int_K \phi_i \phi_l \frac{dQ_{k,l}}{dt} d\mathbf{x} - \sum_{K \in \mathcal{T}_h} \int_K \frac{\partial \phi_i}{\partial x_n} F_{j,n}(\mathbf{q}_h, \nabla_h \mathbf{q}_h + \mathbf{r}([\![\mathbf{q}_h]\!])) d\mathbf{x} \\
&+ \sum_{F \in \mathcal{F}_h} \int_F [\![\phi_i]\!]_n \widehat{F}_{j,n}\left(\mathbf{q}_h^{\pm}, (\nabla_h \mathbf{q}_h + \eta_F \mathbf{r}_F([\![\mathbf{q}_h]\!]))^{\pm}\right) d\sigma \\
&+ \sum_{K \in \mathcal{T}_h} \int_K \phi_i s_j(\mathbf{q}_h, \nabla_h \mathbf{q}_h + \mathbf{r}([\![\mathbf{q}_h]\!])) d\mathbf{x} = 0,
\end{aligned} \tag{4}$$

for $i = 1, \ldots, N_{dof}^K$ and where repeated indices imply summation over the ranges $k = 1, \ldots, m$, $l = 1, \ldots, N_{dof}^K$ and $n = 1, \ldots, d$.

In Eq. (4) $\mathbf{F}$ denotes the sum of the convective and viscous flux functions, and $\widehat{\mathbf{F}}$ the sum of their numerical counterparts. For the former the flux computation is based on the exact solution of the Riemann problem for the artificial compressibility perturbation of the locally 1D inviscid Euler equations, as suggested in Refs. [19,25], while for the latter the BR2 scheme is employed, proposed in Ref. [26] and theoretically analysed in Refs. [27,28].

3. Time discretization

The discrete problem corresponding to Eq. (4) can be written as

$$\widehat{\mathbf{M}} \frac{d\mathbf{Q}}{dt} + \mathbf{R}(\mathbf{Q}) = \mathbf{0}, \tag{5}$$

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