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Nonlinear analysis of beams with rotation gradient dependent potential energy for constrained micro-rotation

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A B S T R A C T

In this study, the weak-form finite element model for bending of beams considering constrained micro-rotation and rotation gradient-dependent potential energy is developed for the moderate rotationcase. The governing equations for a general higher-order beam theory with the von Kármán geometric nonlinearity are derived from the principle of virtual displacements. The formulated finite element model is valid for homogeneous, orthotropic, and functionally graded classical and microstructure-dependent beams. Further, the specialization of the theory to various existing beam theories is also presented. The analytical solution for the simply supported beam for the linear case is also derived. In the numerical examples presented, the stiffening effect due to the consideration of microstructure in the micro-beam is illustrated. The parametric effect of the material length scale on the bending moment and stress is also investigated.

1. Introduction

1.1. Background

There has been surge of research in recent decades in the area of non-classical and non-local continuum mechanics in an attempt to model micro- and nano-sized structures, for example, nematic elastomers, fibrous composites, carbon nanotube-reinforced coatings, granular solids, liquid crystal elastomers, polarization inertia in ferroelectrics, and intrinsic spin in ferromagnetics, to name a few. For such applications, the classical continuum mechanics is inadequate in modeling the true response. In small scale structures, the potential energy due to deformation of the material particles or microstructure, which could be a unit cell in the case of crystalline solid or stiff inclusions in fibrous or granular solid, becomes significant. In these cases, the response depends on several material length scale parameters, which are very small compared to the structural dimensions. In the case of large-scale structures, the ratio of the length scale to the structural dimensions is very small and, therefore, the classical continuum model tend to be adequate for modeling of the response. But as the structural length of the specimen becomes comparable to the characteristic lengths of the material, one must consider non-local and non-classical continuum models.

In general, the continuum where the material points are considered to be undergoing rigid rotations along with displacements during deformation is referred to as the *Cosserat continuum*. For such a solid, there are six degrees of freedom at each material point, namely, three translations and three rotations. Further, the rotation of the material particles or stiff inclusions could be considered as constrained; that is, the microrotation of the material point is the same as the macro rotation at that point and there is no "energy due to rotational mismatch." The inclusion of the additional internal rotational degrees of freedom modifies the balance of angular momentum and gives rise to couple stress and asymmetric stress tensor along with surface tension like force in the case of a solid.

Many researchers (see, e.g., Cosserat and Cosserat, 1909; Truesdell and Toupin, 1960; Toupin, 1962; Mindlin and Tiersten, 1962; Mindlin, 1964; Cemal Eringen and Suhubi, 1964; Suhubl and Cemal Eringen, 1964) have contributed to the development of theory of a Cosserat continuum. In relatively recent times, in the

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case of constrained micro-rotation of material points, Yang et al. (2002) in their modified couple stress theory suggested higher-order moment equilibrium (balance of moment of moments) in the linear framework which results in symmetric couple stress tensor. More recently, Srinivasa and Reddy (2013) has studied the Cosserat continuum in the case of finite rotation and constrained microrotation of material particles. Starting from the physical reasoning, they established the energy dependence on rotation gradient through material frame indifference, and presented the governing equations and boundary conditions for the von Kármán plates and beams in the case of moderate rotations. Upon linearization of the theory, they obtained Mindlin's couple stress theory as a special case. The presence of surface tension-like term is also shown in the work. In the case of beams and plates, they have considered a general quadratic function for the strain energy potential for a general shape and orientation of material particles or inclusions (i.e., not necessarily centro-symmetric), which allows more than one length scale parameter, depending on the orientation of the inclusions in the isotropic matrix of the material.

In the last decade, many papers have appeared on modeling the response of structural elements like beams, plates, and shells, accounting for the length scale effects; they contained parametric studies to determine the length scale effects on bending and vibration response. Park and Gao (2006) and Ma et al. (2008, 2010) have studied the Bernoulli–Euler, Timoshenko, and Reddy–Levinson beam theories in the case of modified couple stress theory. Santos and Reddy (2012) have studied vibrations of beams, while Reddy (2011), Arbind and Reddy (2013), Arbind et al. (2014), and Reddy et al. (2016) studied functionally graded, microstructure dependent beams considering the von Kármán nonlinearity. Gao et al. (2013) studied plates by extending Reddy's third-order plate theory to account for the modified couple stress term in the strain energy functional. Kim and Reddy (2013, 2015) presented analytical and finite element solutions for functionally graded plates with modified couple stress term. In all these studies, the constitutive relation for centro-symmetric material (see Mindlin, 1964) or isotropic Cosserat solid (as termed in Eringen's micropolar theory) is used and rotations of the material particles or inclusions have been assumed to be constrained. For this reason, the curvature tensor is obtained from the deformation field of the matrix material itself. These studies show that the material length scale contributes some extra stiffness to the structure as compared with the conventional theories. Reddy and Srinivasa (2014) has summarized the modified couple stress theory and the rotation gradient dependent theory and formulated finite element models for moderate rotation Bernoulli-Euler and Timoshenko beam theories.

1.2. Present study

The theory suggested by Srinivasa and Reddy (2013), Reddy and Srinivasa (2014) is a generalization of the linear micropolar theory to the case of large constrained microrotation and finite strain for a general class of materials, which requires more than one length scale to characterize a material with an arbitrary microstructure. In the existing literature, the analysis of structures like beams, plates, and shells are based on the constitutive relation in which material points or the small inclusions are centro-symmetric or fully isotropic. And the linear modified couple stress theory has been used in most of the studies mentioned in the previous section for mathematical modeling of structural elements for moderate rotation case.

In the present study, we extend the study of nonlinear response of beams, in view of a broad class of materials with the use of the rotation gradient dependent theory, to account for moderate rotations and strains. We develop a weak-form finite element model of beams with the von Kármán geometric nonlinearity. First, we formulate a general higher-order beam theory based on Taylor's series expansion of the displacement field about the centroidal axis for classical as well as microstructure dependent beams and then specialize it to the case of the Bernoulli–Euler, Timoshenko, and general third-order beam theories. Based on this, we develop a nonlinear weak-form, displacement-based finite element model. We also present the analytical solution for simply supported linear beams to provide a comparison for the finite element solution.

2. Cosserat continuum theory for finite deformation and constrained micro-rotation

Let us consider a body $\mathscr{B}$ in which particle X is at $\mathbf{X}$ in reference frame at time $t = 0$. After deformation, at time t, it occupies position $\mathbf{x}$. Let $\mathbf{F}$ be the deformation gradient and $\boldsymbol{\Theta}$ be the orientation tensor of the directors attached to the material points; then the potential energy can be expressed as (see Srinivasa and Reddy, 2013 for details):

$$\psi = \overline{\psi}(\mathbf{F}, \boldsymbol{\Theta}, \nabla\boldsymbol{\Theta}) \tag{1}$$

where $\nabla\boldsymbol{\Theta}$ is the gradient of the orientation tensor with respect to the reference frame. By applying the principle of invariance under superposed rigid body motion, it can be shown that the potential energy has the following dependence:

$$\psi = \widehat{\psi}\left(\mathbf{C}, \mathbf{R}^T\cdot\boldsymbol{\Theta}, \mathbf{R}^T\cdot\nabla\boldsymbol{\Theta}\right) \tag{2}$$

where $\mathbf{C} = \mathbf{U}^2$ is the right Cauchy–Green stretch tensor and $\mathbf{R}$ is the orthogonal rotation tensor. In the case of fully constrained directors, the orientation tensor can be stipulated as the rotation tensor, and hence the potential energy functional can be expressed as,[1]

$$\psi = \psi\left(\mathbf{U}, \mathbf{R}^T\cdot\nabla\mathbf{R}\right) \tag{3}$$

where $\mathbf{U}$ and $\mathbf{R}$ are symmetric and proper orthogonal tensors, respectively; their variation are $\delta\mathbf{U} = \delta\mathbf{U}^T$ and $\delta\mathbf{R} = \delta\boldsymbol{\Omega}\cdot\mathbf{R}$, where $\delta\boldsymbol{\Omega}$ is a skew-symmetric tensor. Let $\mathbf{f}$ be the body force and $\mathbf{u}$ be the displacement field. Then to obtain the equation of equilibrium, we consider the following lagrangian:

$$\mathbb{L} = \int_{\mathscr{B}} \psi\left(\mathbf{U}, \mathbf{R}^T\cdot\nabla\mathbf{R}\right) - tr\left(\mathbf{P}^T\cdot\mathbf{G}\right) - \mathbf{f}\cdot\mathbf{u}\,\mathrm{d}V, \text{where } \mathbf{G} = \mathbf{R}\cdot\mathbf{U} - \mathbf{F} \tag{4}$$

where $\mathbf{P}$ is the Lagrange multiplier and $\mathbf{G} = 0$ is the constraint condition. In the case of stable equilibrium, the potential energy can be minimized with respect to the displacement field under given constraint conditions, whereas in case of unstable equilibrium (e.g, bucking of beam) or neutral equilibrium of the system, the equation of equilibrium can be obtained by setting the first variation of the above lagrangian equal to zero, that is, from the stationarity condition. Hence, to obtain the Euler–Lagrange equations (i.e., equilibrium equations), we use the stationarity condition $\delta\mathbb{L} = 0$ to obtain the following relations (see Appendix for the detail derivations):

[1] The functions $\overline{\psi}, \widehat{\psi}$ and ψ of the RHS of Eqs. (1)–(3), respectively, represent various functions with different functional dependence of the same physical quantity, that is, the potential energy stored in the body during deformation, ψ. We note that ψ in the RHS of Eq. (3) also represents the functional.