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A hierarchy of simplified constitutive models within isotropic strain gradient elasticity

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Simplified isotropic models of strain gradient elasticity are presented, based on the mutual relationship between the inherent *(dual) gradient directions* (i.e. the gradient direction of any strain gradient source and the lever arm direction of the promoted double stress). A class of *gradient-symmetric* materials featured by gradient directions obeying a *reciprocity relation* and by 4 independent h.o. (higher order) coefficients is envisioned, along with the sub-classes of *hemi-collinear* materials (3 h.o. coefficients, gradient directions in part coincident), *collinear* materials (2 h.o. coefficients, equal gradient directions) and *micro-affine* materials (1 h.o. coefficient, behavioral affinity at micro- and macro-scale, coincident with the Aifantis model). All models comply with the energy positive definiteness conditions. The boundary-value problem for the wide class of gradient-symmetric materials is governed by a set of Poisson–Helmholtz type differential equations almost unaffected by the number of independent h.o. coefficients; instead the boundary conditions carry in, in general, problem-dependent computational difficulties increasing with the number of these coefficients. As an application, gradient-symmetric beam models are discussed. A parallel hierarchy of simplified isotropic models with couple stresses is also presented, in which the novel concept of *rotational volumetric strain gradient* is exploited. A graphical overview on isotropic strain gradient elasticity models is reported. An Appendix is devoted to the concepts of extensional and rotational volumetric strain gradients and to the related pressure-like stresses.

1. Introduction

Strain gradient elastic material models are of increasing interest for research and engineering application purposes. The main features of these models reside in the h.o. (higher order) material coefficients (or constants), which make them capable to predict and describe a number of real phenomena arising from the material inhomogeneities, or from the specimen small scale dimensions (as for example size effects, strain localization, dispersion effects in wave propagation, stress and strain singularities at the crack tips, and the like). Generalized continua theories, and in particular gradient elasticity theories, distinguish themselves from classical theories because of the internal length scale parameter(s) encompassed within the inherent h.o. coefficients, with dimension comparable to the internal length characteristics of the material (as, typically, particle size). This fact makes generalized continuum theories capable to capture small scale phenomena that instead would remain undetected, or detected as stress and strain singularities, within the framework of classical theories. There is an extensive literature concerned with this subject, for which we refer to Germain (1973a, b); Askes and Aifantis (2011); Vardulakis et al. (1996); Fleck and Hutchinson (1997); Lam et al. (2003); dell'Isola et al. (2009); Javili et al. (2013); Auffray et al. (2013) and the literature therein.

A serious obstacle to the exploitation of the above enhanced material models comes up from the absence of an adequate qualitative and quantitative knowledge of the inherent constitutive behavior, apart from a number of experimental studies as Fleck et al. (1994); Ma and Clark (1995); Nix (1989); Stelmashenko et al. (1993); Stolken and Evans (1998); Poole et al. (1996); Lam et al. (2003); Tang and Alici (2011). The latter experimental works provided an important preliminary view on the constitutive behavior of this class of materials, but a rather expensive program of laboratory experiments, paralleled by a complete knowledge of the underlining mechanics, would be necessary. Meanwhile, the formulation of gradient elasticity theories based on a reduced amount of h.o. coefficients (less than five, the number at most

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required in the case of isotropy), constitutes a paramount task, to which the present paper is devoted.

After the pioneering works of Cauchy (1851), Voigt (1887), Cosserat and Cosserat (1909), theories of strain gradient elasticity were advanced by Casal (1961, 1963); Mindlin (1964, 1965); Green and Rivlin (1964a, b); Kröner (1967); Mindlin and Eshel (1968). For the present purposes, of particular interest is the latter paper, where a first strain gradient elasticity theory is advanced for an isotropic material. The starting point of the latter paper is —like in (Mindlin, 1965) where a second strain gradient theory is advanced— a strain energy function depending on the standard strain, $\boldsymbol{\varepsilon}$, as well as on a set of 18 higher order strain measures coinciding with: either the second gradients of the displacements, $\nabla\nabla\mathbf{u}$, (Form I), or the first gradient of the strain, $\nabla\boldsymbol{\varepsilon}$, (Form II), or even the symmetric part (or stretch gradient) $\nabla\boldsymbol{\varepsilon}^S$ (10 components) and the gradient of the continuum rotation $\nabla\boldsymbol{\theta}$ (8 components), (Form III). The latter three forms are shown to be substantially equivalent to one another, whereas Form III is shown to constitute a generalization of the classical couple stress theory, similar to one given by Toupin (1962, 1964), which describes the effects of the stretch gradient, $\nabla\boldsymbol{\varepsilon}^S$, and of the rotation gradient, $\nabla\boldsymbol{\theta}$.

An important contribution to the development of strain gradient elasticity was given by Germain (1973a, b), who addressed the equilibrium problem of first strain gradient continua with microstructure (or micromorphic materials) by means of the principle of virtual power and provided basic guidelines unifying the previous contributions by Toupin (1962, 1964); Mindlin (1964, 1965); Green and Rivlin (1964a, b). A further contribution was given by Wu (1992), who applied the second strain gradient elasticity theory by Mindlin (1965) to surface tension phenomena. Fleck and Hutchinson (1993, 1997), inspired by Toupin (1962) and Mindlin (1964, 1965), addressed isotropic incompressible materials and employed an ad-hoc orthogonal decomposition (previously devised by Smyshlyaev and Fleck (1996)) of the strain gradient tensor carrying in three independent length scale parameters. Fried and Gurtin (2006) formulated a first strain gradient elasticity theory including first velocity gradient inertia and showed the appearance of surface inertia forces in addition to suitably enhanced bulk inertia forces. dell'Isola et al. (2009) provided an extension to finite deformations of a first strain gradient elasticity theory for isotropic materials with the inherent generalized Hooke law. Polizzotto (2012, 2013) used mixed strain gradient/velocity gradient material models to point out some typical boundary effects consisting in the formation of surface layers and edge lines, each of which obeys specific (membrane-like, or rode-like) equilibrium equations in the presence of, respectively, surface and line body forces. Javili et al. (2013) addressed higher order gradient elasticity in the presence of surface and edge line energy densities, and established a sort of hierarchy in the existence of various forms of energy densities. Auffray et al. (2013) addressed strain gradient elasticity in the case of anisotropic materials and provided a matrix representation of the inherent constitutive equations.

Aside the above research line mainly oriented toward questions of theoretical formulations, another parallel research line developed oriented toward application purposes and thus more concerned with simplified material models, that is, featured by a reduced number of h.o. coefficients. Aifantis and co-workers (Aifantis, 1992; Altan and Aifantis, 1992; Ru and Aifantis, 1993), probably inspired by Casal (1961), advanced a material model with only one h.o. coefficient featured by a stress-strain relation in the form of Helmholtz PDEs (partial differential equations) as $\boldsymbol{\sigma}=\mathbf{C}:(\boldsymbol{\varepsilon}-\ell^2\Delta\boldsymbol{\varepsilon})$, where $\mathbf{C}$ is the usual moduli tensor of isotropic elasticity, Δ the Laplacian differential operator and ℓ is a length scale parameter. Polizzotto (2003) and Gao and Park (2007) improved this model by means of a variational procedure leading to the exact form of the extra boundary conditions. In spite of the drastic reduction of the number of h.o. coefficients, the proposed model proved to be capable to predict and describe, more or less adequately, a wide extent of physical phenomena, typically exhibited by real materials, but not detectable by means of the classical model. This fact was probed by a rich amount of applications within both statics and dynamics, for which reference is made to Lazar and Maugin (2005); Lazar et al. (2006); Gao and Ma (2010), along with the review paper by Askes and Aifantis (2011).

Another gradient elasticity model, also motivated by Casal (1961), is the one proposed by Vardulakis and Sulem (1995) in which, beside the strain and strain gradient effects, also their interaction effects are taken into account. This model, anisotropic in nature, is featured, beside a scale parameter like that pertaining to the isotropic model by Aifantis (1992), a director vector specifying the amplitude and the preferred direction in which the mentioned interactions manifest themselves at every point. The model in question has been used for applications within soil mechanics, fracture mechanics, wave propagation and in general to problems in which the surface tension may play a notable role, see e.g. Vardulakis and Sulem (1995); Exadactylos and Vardulakis (1998); Exadactylos et al. (1996); Exadactylos and Vardulakis (2001). A strain gradient isotropic elasticity model for couple stresses featured by only three h.o. coefficients was proposed by Lam et al. (2003) under the assumption that the strain energy is independent of the anti-symmetric part of the rotation gradient. More recently, Gusev and Lurie (2015) formulated a simplified isotropic model of strain gradient elasticity in which an additional symmetry condition is used by which the h.o. coefficients are drastically reduced to only two. Zhou et al. (2016) presented a formulation whereby the sixth-order moduli tensor is transformed into one with only three material constants.

The literature review, not at all exhaustive, presented above shows how the knowledge of the h.o. coefficients is insufficient today. For laboratory experiments and application purposes, but not only, it would be desirable that, within the class of isotropic strain gradient elastic materials, there may exist a hierarchy of constitutive models encompassed between the simplest one by Aifantis (one h.o. coefficient) and the general one (five h.o. coefficients), with every intermediate model being motivated by physically clear assumptions. Indeed, this is the goal of the present paper, in which the formulation based on Form II by Mindlin and Eshel (1968) is followed and the tensor isotropy theory is applied. Starting from the general isotropic model with five h.o. coefficients, a hierarchy of simplified models is derived by enforcing some extra symmetry conditions as the *gradient symmetry* and *gradient collinearity* conditions herein envisioned.

An analogous hierarchy of simplified models involving stretching and couple stresses is also presented, which includes the classic couple stress models studied by Toupin (1962, 1964); Mindlin and Tiersten (1962) along with the simpler models with two constants studied by Koiter (1964) and by Sokolowski (1970). The above formulations show the role played by the (non-standard) "rotational" volumetric strain gradient, equivalent to the skew-symmetric part of the curvature.

A featuring point of the present work is in fact concerned with the *volumetric strain gradient*, which is found to be of two different types, that is, the *extensional* one coinciding with what is usually intended for volumetric strain gradient, and the *rotational* one arising from the rotation gradient.

The paper is organized as follows. In Section 2, some basic notions of tensor isotropy theory are reported for subsequent use, with particular concern to a sixth order tensor. The essential higher order deformation modes of the material are pointed out together with the corresponding response modes. In Section 3, an