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Benchmark solution of the dynamic response of a spherical shell at finite strain

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ABSTRACT

This paper describes the development of high fidelity solutions for the study of homogeneous (elastic and inelastic) spherical shells subject to dynamic loading and undergoing finite deformations. The goal of the activity is to provide high accuracy results that can be used as benchmark solutions for the verification of computational physics codes. The equilibrium equations for the geometrically non-linear problem are solved through mode expansion of the displacement field and the boundary conditions are enforced in a strong form. Time integration is performed through high-order implicit Runge–Kutta schemes. Accuracy and convergence of the proposed method are evaluated by means of numerical examples with finite deformations and material non-linearities and inelasticity.

1. Introduction

Multi-physics codes are used by research institutions to solve important problems in solid and fluid mechanics. Verifying the accuracy of the numerical simulations is thus of paramount importance. One verification technique is to develop analytic solutions to test problems and compare the numerical results of a computational physics code to the analytic solution. Analytic solution typically require simplifying hypothesis to reduce the complexity of the problem. For example, problems involving spherical shells, due to their simple geometry, have been a topic of long-term interest in solid mechanics (Sharpe, 1942; Blake, 1952) (see (Kamm et al., 2008) for a comprehensive overview).

For the dynamic response of solids, when finite strains and material inelasticity are accounted for, common assumptions that are necessary to derive an analytic solution are incompressibility and elastic-perfectly plastic material behavior (Sharpe, 1942; Blake, 1952; Verney, 1968; Xin-Lin, 1994; Cohen et al., 2010; Katzir and Rubin, 2011; Rapoport et al., 2011). Incompressibility and/or particular initial or boundary conditions may be challenging to be modeled in a general purpose solid mechanics code and hence comparison of the computational solution to the analytic one may be difficult. For the aforementioned reasons, this work deals with the formulation of a numerical method that affords high-fidelity and high-accuracy solution. Moreover, due to its flexibility, the developed method lends itself to produce reference solutions for test cases that can be easily reproduced in large computational mechanics codes.

This paper may be regarded as an extension of (Williams et al., 2005; Kamm et al., 2010; Chabaud et al., 2012, 2013a, 2013b, 2015) to the finite deformation and finite strain case. Analogies with the previous works may be found in the truncated modal series expansion of the displacement field, in the method employed to derive the weak form of the governing equations and in the strong enforcement of the boundary conditions at the inner and outer radii. Nonetheless, the method described herein differs substantially from the one used in the small deformation case because eigenfunctions obtained from the homogeneous solution of the small deformation case are not solutions of the equilibrium equations in presence of finite deformations. Therefore, a Chebyshev polynomial expansion is used in place of the truncated series of eigenfunctions. Chebyshev polynomials also offer numerical advantages because spatial integration is exact whereas integration of eigenfunctions, based on Bessel functions, poses severe challenges since numerical cancellation reduces integration accuracy for high order modes (Iserles et al., 2006; Chen, 2015).

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In the geometrically linear case (Williams et al., 2005; Kamm et al., 2010; Chabaud et al., 2012, 2013a, 2013b, 2015) an analytic expression for the evolution of the kinematic variables is derived. When material inelasticity is accounted for (Williams et al., 2005; Chabaud et al., 2013b, 2015), the inelastic contributions are collected within the eigen-stress vector and an iterative procedure is used to compute the converged equilibrium solution for each time increment. Although in the finite deformation case an analytic solution for the evolution of acceleration, velocity and displacement does not exist, it would be possible to extend the approach used for the small deformation to the geometrically non-linear case. The eigen-stress fields may collect the non-linear contributions arising from material inelasticity and geometric non-linearities and an iterative solver may be used to obtain the converged equilibrium solution. Although viable, this approach is abandoned in favor of numerical time integration which affords higher accuracy, higher convergence order and does not require the computation of the eigen-functions based on the Bessel functions.

An implicit Runge–Kutta (IRK) scheme is chosen to evolve the kinematic variables: accelerations and velocity increments during a time-step are expressed as functions of the displacement increment and the resulting non-linear system of equations in solved, for the displacement variables, with a Newton-Raphson iterative method. If a diagonal implicit Runge–Kutta (DIRK) is used, the time integration scheme in (Ellsiepen and Hartmann, 2001; Hartmann, 2002) is recovered. Amongst the possible IRK schemes based on collocation methods (Hairer et al., 1993), the stiffly accurate, A-, B- and L-stable, Lobatto IIIC scheme is employed. Due to the displacement field interpolation and to the coupling introduced by finite deformations, the structure of the linearized system is dense, unlike the one obtained from finite elements. Therefore, the additional coupling introduced by the considered IRK scheme, does not modify the structure of the linearized system.

The paper is structured by introducing linear momentum conservation for a spherical shell with spherical symmetry conditions in Section 2. The approximation of the displacement field and the weak form of the problem are presented in Section 3. In Section 4, the time integration scheme employed for the evolution of the kinematic variables is introduced. Subsequently, several hyperelastic material constitutive models are described in Section 5. Numerical simulations involving homogeneous isotropic spherical shells showing elastic and elasto-plastic behavior are carried out in Section 6 to assess the performance of the present formulation in the context of dynamic analysis. Finally, a summary and concluding remarks are presented in Section 7.

2. Problem statement

A homogeneous spherical shell with inner radius r_i and outer radius r_o is considered. A spherical coordinate system is defined at the center of the shell and $\boldsymbol{u} = [u_r, u_\theta, u_\phi]^T$ is the displacement vector of every point belonging to the shell. Moreover, the initial displacement and velocity of a point of the shell are $\boldsymbol{u}_0$ and $\dot{\boldsymbol{u}}_0$ respectively. The domain's boundary, $\partial\mathscr{B}_0$, consists of two disjoint subsets, $\partial_u\mathscr{B}_0$ and $\partial_\sigma\mathscr{B}_0$, where essential (Dirichlet) and natural (Neumann) boundary conditions, respectively, are imposed. It should be observed that, unlike (Williams et al., 2005; Kamm et al., 2010; Chabaud et al., 2012, 2013a, 2013b, 2015), in the present work the boundary conditions are defined in accordance with (Malvern, 1969). The problem is governed by the principle of conservation of linear momentum, and can be stated as follows (Malvern, 1969):

$$\nabla_0 \cdot \mathbf{P} + \boldsymbol{b} = \rho_0 \ddot{\boldsymbol{u}} \quad \text{in } \mathscr{B}_0, \tag{1}$$

$$\boldsymbol{u} = \overline{\boldsymbol{u}} \quad \text{on } \partial_u\mathscr{B}_0, \tag{2}$$

$$\mathbf{P}\cdot\boldsymbol{n} = \overline{\boldsymbol{h}} \quad \text{on } \partial_\sigma\mathscr{B}_0, \tag{3}$$

$$\boldsymbol{u} = \boldsymbol{u}_0 \quad \text{at } t = 0, \tag{4}$$

$$\dot{\boldsymbol{u}} = \dot{\boldsymbol{u}}_0 \quad \text{at } t = 0, \tag{5}$$

where $\mathbf{P}$ and ρ_0 respectively denote the first Piola-Kirchhoff stress tensor and the mass density of the material in the initial configuration. Standard transformations (Malvern, 1969) are used to map between $\mathbf{P}$ and the second Piola-Kirchhoff stress, $\mathbf{S}$, and between $\mathbf{P}$ and the Cauchy stress, $\boldsymbol{\sigma}$.

Assuming spherical symmetry $u_\theta = u_\phi \equiv 0$, $S_{\theta\theta} = S_{\phi\phi}$, and $S_{r\theta} = S_{r\phi} = S_{\theta\phi} \equiv 0$, the problem becomes one-dimensional and, for the sake of readability, the following simplified notation in used

$$u \equiv u_r, \quad S_r \equiv S_{rr}, \quad S_\theta \equiv S_{\theta\theta}.$$

Furthermore, a total Lagrangian approach is employed and all the derivatives are referred to the initial configuration (i.e. $\frac{\partial_0}{\partial_0 r} = \frac{\partial}{\partial r}$). The conservation of linear momentum for a spherical shell in Cartesian coordinates is hence written as

$$\rho_0 \ddot{u} = \frac{2}{r}(S_r - S_\theta) + \frac{\partial S_r}{\partial r}\left(1 + \frac{\partial u}{\partial r}\right) + S_r \frac{\partial^2 u}{\partial r^2} - \frac{2}{r^2} S_\theta u + \frac{2}{r} S_r \frac{\partial u}{\partial r} \quad \text{in } \mathscr{B}_0, \tag{6}$$

$$u = \overline{u} \quad \text{on } \partial_u\mathscr{B}_0, \tag{7}$$

$$P_r = \overline{h} \quad \text{on } \partial_\sigma\mathscr{B}_0, \tag{8}$$

$$u = u_0 \quad \text{at } t = 0, \tag{9}$$

$$\dot{u} = \dot{u}_0 \quad \text{at } t = 0, \tag{10}$$

where the second Piola-Kirchhoff stress at every point is obtained from the material constitutive model and where the body force term has been neglected.

3. Solution method

The method employed to solve the balance of linear momentum in Eq. (6) may be split in three major steps: (i) define an appropriate modal basis for the approximation of the displacement field (N modes), (ii) the equilibrium equation (6) is multiplied by the first $N-2$ modes and integrated over the shell's volume to obtain $N-2$ equations and (iii) two additional equations necessary to compute the N modal coefficients are derived directly from the boundary conditions Eqs (7) and (8).

The displacement field is assumed to be given as truncated series expansion of N modes (Williams et al., 2005), collected within the $\boldsymbol{\Psi}$ vector,

$$u = \frac{1}{\sqrt{r}} \sum_{i=1}^{N} q_i \Psi_i = \sum_{i=1}^{N} q_i \overline{\Psi}_i, \tag{11}$$

where the vector of unknown coefficients, $\boldsymbol{q}$, defines the modes' amplitude and the $1/\sqrt{r}$ factor is introduced for consistency with (Williams et al., 2005). Hereafter, a polynomial basis built with the