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Finite-strain low order shell using least-squares strains and two-parameter thickness extensibility

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ABSTRACT

We present a thickness-extensible finite strain quadrilateral element based on least-squares in-plane shear strains and assumed transverse-shear strains. At each node, two thickness parameters are connected to the constitutive laws by a linear system. The zero out-of-plane normal stress condition is satisfied at the constitutive level using the normal strain as unknown in all integration points. Assumed in-plane strains based on least-squares are introduced as an alternative to the enhanced-assumed-strain (EAS) formulations and, contrasting with these, the result is an element satisfying *ab-initio* both the in-plane and the transverse Patch tests. There are no additional degrees-of-freedom, as it is the case with EAS, even by means of static condensation. Least-squares fit allows the derivation of invariant finite strain elements which are shear-locking free and amenable to be incorporated in commercial codes. With that goal, we use automatically generated code produced by AceGen and Mathematica. Full assessment of the element formulation and the two-parameter thickness variation methodology is accomplished. Alternative thickness variation algorithms are tested. All benchmarks show very competitive results, similar to the best available enriched shell elements.

1. Introduction

Hyperelastic and finite strain plasticity constitutive models require precise kinematics with structural elements: length-preserving interpolated directors and thickness variation are essential ingredients. In addition, non-linear shell simulations with finite elements (cf. (Areias et al., 2013a)) are demanding with respect to numerical efficiency, Newton iteration robustness and mesh distortion insensitivity. This is relevant in the edge-based algorithms recently proposed (Areias et al., 2013c) when applied to quadrilaterals. Many of the intricate element formulations, such as enhanced-assumed-strain (EAS (Areias et al., 2003)), hybrid stress, discrete Kirchhoff (DK, cf. (Areias et al., 2005)), are suitable for smooth problems where the mesh distortion sensitivity is not a crucial ingredient and governing equations do not contain discontinuities. In addition, costs associated with convergence difficulties and static condensation, specifically with EAS, can be high. The same applies to meshless methods in shells (cf. (Rabczuk and Areias, 2006; Rabczuk et al., 2007)): geometrically complex shell problems pose difficulties with meshless methods.

Thickness variation is an important attribute for finite strains, and in shells it is a consequence of kinematics and constitutive laws. Often, either closed-form solutions exist, as in Hookean elasticity or certain hyperelastic materials (cf. (Bonet and Wood, 2008)) or a two-level iteration is used. We here take a more direct approach:

- Two thickness parameters are used, representing distances between the shell faces and the reference surface.
- A simultaneous iteration scheme for the normal strain, constitutive stress correction and plastic multipliers is proposed. The consistent tangent modulus is derived.

Coupling of thickness variation and zero normal stress condition is not a new attribute in shell analysis. The first systematic

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Nomenclature

$\boldsymbol{\alpha}$	assumed strain unknown parameters
$\Delta\gamma$	plastic multiplier increment
κ_s	shear correction parameter
$\mu^\star$	scaling factor for the plastic multiplier
ν	Poisson coefficient
ξ	curvilinear coordinates
σ_{ij}, $\boldsymbol{\sigma}$	Cauchy stress tensor
$\phi(\Delta\check{\boldsymbol{S}}_{ab})$	yield function
∇_a	gradient with respect to configuration Ω_a
a_n	lower thickness parameter
b_n	upper thickness parameter
b_i, $\boldsymbol{b}$	body force
$\boldsymbol{C}_{ab}$	right Cauchy-Green tensor using Ω_a and Ω_b as equilibrium and reference configurations, respectively
$\boldsymbol{C}^\star_{ab}$	mixed-variant right Cauchy-Green tensor
$\mathscr{C}$	tangent modulus
$\mathscr{C}_{\text{linear}}$	linear tangent modulus
$\mathscr{C}_{\star\text{linear}}$	reduced linear tangent modulus
$\boldsymbol{d}_a$	unit director for configuration Ω_a
E	elasticity modulus
$\boldsymbol{E}_{ab}$	green-Lagrange strain using Ω_a and Ω_b as equilibrium and reference configurations, respectively
$\boldsymbol{E}^\star_{ab}$	mixed-variant Green-Lagrange strain
$[\mathbf{E}_{ab}]_{33}$	strain for the thickness direction
$\tilde{\mathbf{E}}$	assumed strain (Voigt form)
$\tilde{\mathbf{E}}_I$	out-of-plane assumed strains
$\tilde{\mathbf{E}}_{II}$	in-plane assumed strains
$\boldsymbol{F}$	deformation gradient
$\boldsymbol{h}_c$	vector of constitutive thickness parameters
$\boldsymbol{h}_N$	vector of nodal thickness parameters
H_0	initial thickness
H_b	thickness
$H^\star_b$	effective thickness function
$\boldsymbol{I}$	identity matrix
J	determinant of the deformation gradient
$\boldsymbol{J}_a$	Jacobian of configuration Ω_a
L_w	least-squares function
$m_{13A}, m_{13B}, m_{23C}, m_{23D}$	assumed-natural strain components at points A,B,C and D, respectively
	$\boldsymbol{M} = \int_{\Omega_b} \boldsymbol{Q}^T\boldsymbol{Q}\,\mathrm{d}\Omega_b$
$\boldsymbol{m}_{aa}$	matrix of covariant metric coefficients for configuration Ω_a
$\mathbf{n}$	flow vector
$\boldsymbol{N}_h$	thickness interpolation matrix
$N_K(\xi_1,\xi_2)$	Nodal shape functions
$\boldsymbol{Q}$	assumed strain interpolation function
$\boldsymbol{r}_a$	mid-surface position
$\boldsymbol{S}$	second Piola-Kirchhoff stress tensor
$\boldsymbol{S}_b$	second Piola-Kirchhoff stress tensor using Ω_b as reference configuration
$\boldsymbol{S}_{ab}$	second Piola-Kirchhoff stress tensor using Ω_a and Ω_b as equilibrium and reference configurations, respectively
$\mathbf{S}^{'}_{ab}$	second Piola-Kirchhoff stress tensor in the local frame (Voigt form)
$\Delta\check{\boldsymbol{S}}^5_{ab}$	reduced Voigt form of the constitutive part of the stress
$\Delta\check{\boldsymbol{S}}_{ab}$	constitutive part of the stress
$\boldsymbol{T}_1, \boldsymbol{T}_5$	stress projection transformation
$\boldsymbol{T}_S(\boldsymbol{R})$	Stress transformation matrix
$\boldsymbol{T}_E(\boldsymbol{R})$	strain transformation matrix
$\boldsymbol{T}_{E\star}$	strain transformation matrix for the fixed frame
$\overset{\triangle}{W}_{\text{ext}}$	virtual power of external forces
x_{a_i}, $\boldsymbol{x}_a$	coordinates of a point in configuration Ω_a

treatment was performed by Hughes and Carnoy (Hughes and Carnoy, 1983) for hyperelasticity. Although the reference surface position was parametrized in that paper, it was considered user-defined data. Crucially, the normal strain was integrated through the thickness coordinate to update the thickness parameter. Bending behavior was not inspected in detail. More recently, Sussman and Bathe (cf. (Sussman and Bathe, 2013)) have indicated the need for use of two thickness parameters for performing genuine finite strain shell simulations. They also introduced warping degrees-of-freedom to represent independent motion of "top" and "bottom" directors. We make use of a simplified version of Sussman and Bathe kinematics, with thickness strains being piecewise constant instead of piecewise linear as proposed in their work. In addition, contrasting with their work (which makes use of plane strain constitutive laws), we use static condensation at the constitutive level to obtain a zero normal stress constitutive law. In the present paper we only consider initially uniform thickness, although extension to initial varying thickness is straightforward by use of the shape functions. Compared with other thickness-extensible finite strain shell formulations, for example Klinkel, Gruttmann and Wagner (Klinkel et al., 2008) where besides displacements, strains and stresses are interpolated in a 3-field formulation, the present computational costs and formulation intricacy are lower. Recent corotational quadrilateral elements, cf. (Li et al., 2013) are also more intricate. Non-corotational triangles have been recently introduced with assumed-natural strains (ANS) by Bathe's group, cf. (Lee et al., 2014; Jeon et al., 2014, 2015).

In terms of shell formulation, starting with a mixed functional (displacement field, director field, components of the local Cauchy-Green tensor in covariant/contravariant coordinates and the corresponding stress-like Lagrange multipliers), we discretize the resulting Euler-Lagrange equations making use of appropriate shape functions. A complete testing program is then performed. The set of obstacle problems for shells are the classical plate and shell benchmarks and extensions to finite strains. Besides thickness variation, it is important to test elements in finite strains since some instabilities have been found in the past (see (Crisfield and Peng, 1996) for a report with the Morley-based shell). In terms of quadrilateral shell element technology, some important works should be mentioned. A milestone in the removal of transverse shear locking was achieved with the assumed natural strain (ANS) technique in 1984 and 1986 (Dvorkin and Bathe, 1984; Park and Stanley, 1986). A decade earlier, in-plane bending locking was solved in 1973 by the Wilson Q6 element (Wilson et al., 1973), with several ulterior corrections. For undistorted meshes, convergence rate of the results is established regardless of the incomplete higher order terms in the polynomials (see the book by Belytschko and coworkers (Belytschko et al., 2000)) and these higher order terms only contribute to stability and coarse-mesh accuracy. More recently, Ko, Lee and Bathe (Ko et al., 2016) introduced in-plane assumed natural strains as an alternative to the Q6 enrichment in shells. This follows an important work by Sussman and Bathe (Sussman and Bathe, 2014) where it was proved that EAS elements are inherently unstable in large strain conditions. In addition, mesh