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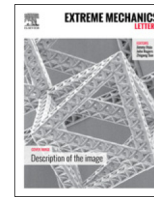

Dynamic homogenization theory for nonlocal acoustic metamaterials

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ABSTRACT

We present a homogenization method for periodic acoustic composites based on the Plane Wave Expansion (PWE) method. We show that the description of periodic acoustic composites needs constitutive parameters which depend on frequency and wavenumber, meaning that the effective material is resonant and nonlocal. Also, an anisotropic mass density and an additional constitutive parameter, called the Willis term in analogy to its counterpart in elasticity, are found. Numerical calculations compare the present method with the traditional multiple scattering method, showing a good agreement between both theories. However, the method presented here overcomes the limitations of the multiple scattering method, where the incorporation of anisotropy and non-locality implies solving the scattering problem of anisotropic objects with additional boundary-conditions. A final example showing the importance of nonlocal effects is provided. This work shows that acoustic metamaterials are nonlocal materials in general, and provides a tool for the proper modeling of the nonlocal constitutive parameters.

1. Introduction

The study of the propagation of acoustic waves through complex media has received a new insight within the last ten years with the advent of the so called "acoustic metamaterials" [1]. These structures consist essentially in arrangements (periodic or not) of resonators designed in such a way that, when the wavelength of the acoustic wave is larger than the typical distance between scattering units, the structure behaves as an effective material with frequency-dependent constitutive parameters. Then, metamaterials present resonant constitutive parameters so that they can behave like materials with negative mass density [1], bulk modulus [2] or both simultaneously [3–5]. Additionally, it was also shown that acoustic metamaterials required for their proper description an anisotropic mass density, even in the quasi-static limit [6]. These unusual constitutive parameters offer as well a wide variety of applications, like hyperlenses [7,8], cloaking devices [6,9] or omnidirectional absorbers [10,11].

Describing micro-structured materials at the macro-scale by means of homogenization methods not only shortens the simulation time, but it also facilitates the design of metamaterials by a description in terms of effective parameters. Additionally, it also gives new ways to describe the physical behavior of heterogeneous micro-structured media. However, the development of mathematical tools for the dynamical description of these composites is a complex task [12]. Although homogenization theories for fluid or solid composites in the quasi-static limit are well known [13,14], finite-frequency methods have been recently developed based on the coherent potential approximation [15,16] or the multiple scattering theory [17,18], which have allowed the inclusion of a resonant-like behavior of the constitutive parameters. Recent methods allow also for the description of composites even in the high frequency limit [19], where the discussion about the meaning of the constitutive parameters is even more complex.

In the previously referenced works, the underlying idea for the homogenization of metamaterials is that a combination of materials gives a composite with frequency-dependent constitutive parameters. However, the pioneering work of Willis [20] on the homogenization of elastic composites revealed that in the dynamic regime, composites could change the nature of the constituent materials, in the sense that new constitutive parameters would emerge as a consequence of the averaging process. In this context, equations for spatially averaged fields are of the Willis form [21–23]: the mass density becomes tensorial and momentum and strain are coupled by means of the so called "Willis tensor", which is a new constitutive parameter not necessarily found in the individual materials forming the composite. The exact expressions of frequency–wavenumber dependent effective parameters were first

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derived by Willis for a periodic laminate [22] in term of the Green's function of the media. Efforts have then been made to find analytical and easier calculable expressions, first in one-dimensional [24–26] periodic composites, and later in three-dimensional [27–29] periodic structures. The philosophy of Willis work is combined with the Bloch–Floquet decomposition and the Plane Wave Expansion method [28,29], the monodromy matrix formalism [24,30], or the comparison with a reference medium [27].

In this context, we present a homogenization theory, based on the Plane Wave Expansion (PWE) method, which defines a set of generalized (non-local) effective parameters for periodic acoustic composites. As mentioned before, it is shown that both the effective mass density and bulk modulus of these composites are frequency dependent and non-local, with the additional complexity that the mass density is also anisotropic. Moreover, it is found that an additional constitutive parameter, called the Willis term in analogy to its counterpart in elasticity, is needed for their description. It will be introduced here but not deeply analyzed, since the present work focuses its attention on the non-local properties of the effective parameters.

The paper is organized as follows: Section 2 outlines the homogenization model used to calculate the constitutive parameters of the propagation. Section 3 compares the present method with the classical multiple-scattering homogenization method. Section 4 shows a numerical example consisting of a rectangular lattice of cylinders which behaves as an anisotropic metamaterial with non-local effective parameters. Finally, the conclusions and the perspectives of this work are reported in Section 5.

2. Homogenization theory for periodic acoustic composites

We present a homogenization theory for sonic crystals based on the PWE method [31–33], and similar in methodology to that previously developed for phononic crystals [29]. The starting point is the equation of motion for an inhomogeneous fluid, in which a harmonic time dependence of frequency ω has been assumed,

$$\nabla \cdot [\rho^{-1}(\mathbf{r})\nabla P(\mathbf{r})] = -B^{-1}(\mathbf{r})\omega^2 P(\mathbf{r}), \tag{1}$$

with $\rho(\mathbf{r})$ and $B(\mathbf{r})$ being the position dependent mass density and bulk modulus, respectively, and $P(\mathbf{r})$, the pressure field. For a homogeneous medium $\rho(\mathbf{r}) = \rho_b$ and $B(\mathbf{r}) = B_b$, the dispersion relation is obtained by assuming plane wave propagation with wave vector $\mathbf{k} = k\mathbf{n}$ (with $\mathbf{n}$ the direction of the propagation),

$$k^2 = \omega^2 \rho_b / B_b. \tag{2}$$

In an infinite periodic structure, $\rho(\mathbf{r})$ and $B(\mathbf{r})$ are periodic functions of $\mathbf{r}$, then they can be expanded in a Fourier series on the reciprocal lattice vector $\mathbf{G}$. Under this periodicity condition, Bloch theorem states that the pressure field can be expanded as

$$P(\mathbf{r}) = e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \sum_{\boldsymbol{G}} P_{\boldsymbol{G}} e^{i\boldsymbol{G}\cdot\boldsymbol{r}}, \tag{3}$$

which inserted into the wave equation becomes

$$\sum_{\mathbf{G}'} \sum_i (\mathbf{k}+\mathbf{G})_i \rho^{-1}_{\mathbf{G}-\mathbf{G}'} (\mathbf{k}+\mathbf{G}')_i P_{\mathbf{G}'} = \omega^2 \sum_{\mathbf{G}'} B^{-1}_{\mathbf{G}-\mathbf{G}'} P_{\mathbf{G}'}, \tag{4}$$

with $\rho^{-1}_{\boldsymbol{G}}$ and $B^{-1}_{\boldsymbol{G}}$, the Fourier components of the inverse of the mass density and inverse of the bulk modulus, respectively. The index i indicates a sum over the three components of $\mathbf{k}$, $\mathbf{G}$ and $\mathbf{G}'$ vectors. In matrix form, Eq. (4) is expressed as

$$\sum_{\mathbf{G}'} M_{\mathbf{G}\mathbf{G}'} P_{\mathbf{G}'} = \omega^2 \sum_{\mathbf{G}'} N_{\mathbf{G}\mathbf{G}'} P_{\mathbf{G}'} \tag{5}$$

with

$$M_{\mathbf{G}\mathbf{G}'} = \sum_i (\mathbf{k}+\mathbf{G})_i \rho^{-1}_{\mathbf{G}-\mathbf{G}'} (\mathbf{k}+\mathbf{G}')_i \tag{6}$$

and

$$N_{\mathbf{G}\mathbf{G}'} = B^{-1}_{\mathbf{G}-\mathbf{G}'}. \tag{7}$$

Eq. (5) is a generalized eigenvalue equation, which for a given $\boldsymbol{k}$ returns a set of eigenfrequencies ω. The eigenfrequencies yield the band structure or dispersion curve $\omega = \omega(\boldsymbol{k})$ of the periodic medium. The description of the phononic crystal as a homogeneous material is made by assuming that the effective pressure field propagates as a purely Bloch wave, that is

$$P_{\text{eff}} = \langle P \rangle e^{i\boldsymbol{k}\cdot\boldsymbol{r}}, \tag{8}$$

with $\langle P \rangle$ being the spatial average of the pressure field. Then, finding an equation for $\langle P \rangle$ and its relationship with ω and $\boldsymbol{k}$ will give a definition of an effective medium and will allow an identification of the different constitutive parameters. Since the material is periodic, the average of the pressure field will be equal to the average in the unit cell, so that from Eq. (3) we find that

$$\langle P \rangle = P_{\mathbf{G}=\mathbf{0}}. \tag{9}$$

The average of the pressure field is then given by the $\mathbf{G} = \mathbf{0}$ component of $P_{\mathbf{G}}$, which can be obtained by expressing Eq. (5) as

$$\omega^2 N_{\mathbf{00}} P_{\mathbf{0}} + \omega^2 \sum_{\mathbf{G}'\neq\mathbf{0}} N_{\mathbf{0}\mathbf{G}'} P_{\mathbf{G}'} = M_{\mathbf{00}} P_{\mathbf{0}} + \sum_{\mathbf{G}'\neq\mathbf{0}} M_{\mathbf{0}\mathbf{G}'} P_{\mathbf{G}'} \tag{10a}$$

$$\omega^2 N_{\mathbf{G}\mathbf{0}} P_{\mathbf{0}} + \omega^2 \sum_{\mathbf{G}'\neq\mathbf{0}} N_{\mathbf{G}\mathbf{G}'} P_{\mathbf{G}'} = M_{\mathbf{G}\mathbf{0}} P_{\mathbf{0}} + \sum_{\mathbf{G}'\neq\mathbf{0}} M_{\mathbf{G}\mathbf{G}'} P_{\mathbf{G}'}. \tag{10b}$$

Hereafter, it is considered that matrix elements labeled with $\mathbf{G}$ do not include the term $\mathbf{G} = \mathbf{0}$, which is extracted from the above decomposition. We can now solve from the second equation for $P_{\mathbf{G}}$,

$$P_{\mathbf{G}'} = -\sum_{\mathbf{G}\neq\mathbf{0}} \chi_{\mathbf{G}'\mathbf{G}}(\omega, \mathbf{k})(M_{\mathbf{G}\mathbf{0}} - \omega^2 N_{\mathbf{G}\mathbf{0}}) P_{\mathbf{0}} \tag{11}$$

where

$$\chi_{\mathbf{G}'\mathbf{G}}(\omega, \mathbf{k}) = (M_{\mathbf{G}'\mathbf{G}} - \omega^2 N_{\mathbf{G}'\mathbf{G}})^{-1}_{\mathbf{G}'\mathbf{G}}, \tag{12}$$

and insert it into the first one (Eq. (10a)), obtaining the following equation

$$\Bigg[\omega^2 N_{\mathbf{00}} - \sum_{\mathbf{G},\mathbf{G}'\neq\mathbf{0}} \omega^2 N_{\mathbf{0}\mathbf{G}'} \chi_{\mathbf{G}'\mathbf{G}} (M_{\mathbf{G}\mathbf{0}} - \omega^2 N_{\mathbf{G}\mathbf{0}}) - M_{\mathbf{00}} + \sum_{\mathbf{G},\mathbf{G}'\neq\mathbf{0}} M_{\mathbf{0}\mathbf{G}'} \chi_{\mathbf{G}'\mathbf{G}} (M_{\mathbf{G}\mathbf{0}} - \omega^2 N_{\mathbf{G}\mathbf{0}})\Bigg] P_{\mathbf{0}} = 0. \tag{13}$$

Eq. (13) is formally the same equation as (5), however it is not an eigenvalue equation, but a secular equation for $P_{\mathbf{0}}$ similar to Eq. (2), where the solutions $\omega = \omega(\mathbf{k})$ are obtained from the zeros of the function Γ,

$$\Gamma = \omega^2 N_{\mathbf{00}} - \sum_{\mathbf{G},\mathbf{G}'\neq\mathbf{0}} \omega^2 N_{\mathbf{0}\mathbf{G}'} \chi_{\mathbf{G}'\mathbf{G}} (M_{\mathbf{G}\mathbf{0}} - \omega^2 N_{\mathbf{G}\mathbf{0}}) - M_{\mathbf{00}} + \sum_{\mathbf{G},\mathbf{G}'\neq\mathbf{0}} M_{\mathbf{0}\mathbf{G}'} \chi_{\mathbf{G}'\mathbf{G}} (M_{\mathbf{G}\mathbf{0}} - \omega^2 N_{\mathbf{G}\mathbf{0}}). \tag{14}$$

In the above equation the coefficients are in general functions of both ω and $\mathbf{k}$, what makes it less suitable for band structure calculation than Eq. (5) but more suitable for the description of the sonic crystal as an effective material. Indeed, we can see that the