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Equivalent property between the one-half order and first-order shear deformation theories under the simply supported boundary conditions

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A B S T R A C T

It was verified that if the boundary conditions for all edges of a plate or both ends of a beam are assumed to be simply supported the theoretical framework of the one-half order shear deformation plate or beam theory with total deflection w being assumed as the sum of the bending and shearing deflections w_b and w_s results in being equivalent to that of the Mindlin plate theory or the Timoshenko beam theory. Based on its equivalent property, exact frequency relationships between the classical Kirchhoff plate and the above two equivalent shear deformable plate theories were deduced for general polygonal plates with all edges simply supported, and in addition an approximate frequency relationship which has a very simple form and almost enough accuracy for practical use was obtained based on the series-type synthetic-frequency method.

1. Introduction

Moderately thick plates and beams or laminated composites are widely used for various structures. Although there exists a vast amount of studies based on the classical thin plate or beam theories represented by the Kirchhoff plate or Euler-Bernoulli beam, those analysis models cannot be used when the plate or beam thickness becomes relatively large because in thick or laminated structures the shear deformation effect in addition to the rotary inertia effect cannot be neglected. To overcome this, many shear deformable theories have been presented to date involving higher-order shear deformation theories which don't require the so-called shear correction factor (e.g., Reddy [1,2]; Soldatos [3]; Hanna and Leissa [4]; Nguyen et al. [5]; Nguyen et al. [6]; Yahia et al. [7]; Mahi et al. [8]; Bourada et al. [9]), but even though many accurate higher-order theories are available, first-order shear deformation theories represented by the Timoshenko beam [10,11] and Mindlin plate [12] models, which will be designated as FSDT henceforth in this paper, still continue to be the focus of much research because of their simplicity of analysis. In addition, there exists one more type of shear deformable theory with inherent simplicity of analysis, in which assumed variables can be reduced by introducing an idea of partitioning the total deflection w into the bending and shearing deflections w_b and w_s (e.g., Shimpi [13]; Senjanovic et al. [14]; Endo and Kimura [15]).

Concerning those deformation concepts in addition to FSDTs, a recent review was given by Endo [16], in which historical survey on the physical recognition of deformation was carried out, including the so-called corrected classical theory advocated by Donnell [17]. In this theory, the bending and shearing deflections w_b and w_s are recognized as distinguishable physical entities and the separately obtained shearing deflection w_s is simply added to the bending deflection w_b to give the total deflection w using a non-deductive approach.

Based on the above-mentioned deformation concept as $w = w_b + w_s$, Shimpi et al. presented two new first-order shear deformation plate theories [18], and Thai and Choi presented a simple first order shear deformation theory for plates [19,20], In addition, the author himself presented an "alternative" first order shear deformation concept and applied it to the bending and vibration problems for beams, plates and cylindrical shells [21]. And in the discussions of the paper, he concluded that its analysis model is to be regarded as a refined mathematical generalization of the corrected classical theory. Further, as an extended study of this paper, he examined its concept more carefully in his recently published paper [22], and he proposed to call its analysis model ' one-half order shear deformation theory,' which will be designated as "present" plate or beam theory in the coming text in this paper.

The present theory has some advantages that (1) being consistent with the corrected classical theory in the sense that the bending and shearing deflections w_b and w_s could be obtained uniquely as distinguishable physical entities, (2) being useful for developing an FEM element formulation free from shear locking and (3) reduction of assumed variables by partitioning the total deflection w into the bending and shearing deflections w_b and w_s, which becomes especially effective when we consider the plate or beam stretching effects in the analyses of, for example, laminated composites (e.g., Nguyen et al. [5]; Thai and Choi [19]) or functionally graded structures(e.g., Nguyen et al. [6]; Thai and Choi [20]) necessarily involving in-plane extension displacement vari-

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ables. Here, it is to be noted that concerning functionally graded material (FGM) plates Hebali et al. [23] and Belabed et al. [24] presented higher order shear deformation theories in which the transverse displacement is divided into bending, shear and besides thickness stretching parts and the number of unknowns and governing equations is successfully reduced. And, Belabed et al. state in their paper [24] that "the assumption of transverse strain $\varepsilon_z = 0$ is appropriate for thin or moderately thick FGM plates, but is inadequate for thick FGM plates." And therefore, in the present paper we will confine to "moderately thick" structural analysis models.

The naming as "one-half order" is owing to the fact that in the present theory only the half categories of shear deformations, which involve also in-plane rotational shearing in addition to transverse shearing, can be expressed explicitly in terms of fundamental variables in contrast to the traditional FSDT, which can take into account both shearing effects fully. Particularly, it should be noted that the traditional first-order plate theory can take into account the in-plane rotational effect in the $x-y$ plane of a plate element with the aid of a rotational potential ψ (Endo [21]), whereas the present plate theory cannot involve its effect in principle. As suggested in the authors previous papers [15,21], if we consider the boundary value problems such as SCSC, SSSC, SCSF, SFSF or SSSF plates in addition to SC beam (where, S, C and F indicate simply-supported, clamped and free boundary conditions), the deviation of the results of the present theory from those of the traditional first-order shear deformation theories becomes large, which is considered to be owing to the lack of rotational potential ψ, at least for a plate problem.

In this sense, we are safe to say that FSDT is the lowest order shearing model which can successfully describe full behaviors of shear deformable thin-walled structures, though it necessarily needs an assumption of the shear correction factor. Thus, the order of superiority of the structural accuracy of the present model in comparison with the three dimensional elasticity theory is to be placed midway between the classical and traditional FSDTs.

However, to the best of the author's experience, almost equivalent frequency or bending solutions could be obtained between the present and FSDTs [15,16,21] at least in such the cases that all edges/ends of a rectangular plate or straight beam are simply supported. In addition, although the illustrative computational results other than the cases for simply supported boundary conditions are comparatively few in the concerned literature, Thai and Choi [19,20], Houari et al. [25] and Bouderba et al. [26] reported that the accuracy of their simple shear deformation theories is almost the same as that of the traditional first-order shear deformation theories at least for simply supported plates. However, the above knowledge is based on the numerical comparison of the calculated results, and an in-depth examination of its nature based on the theoretical framework itself never exists to date.

Considering the above, let us consider its possible reasons from the viewpoint of theoretical framework characteristics, below.

If we eliminate the rotation ϕ of a beam element in the governing equations for the Timoshenko beam theory (Timoshenko [10]; Endo [22]), we can obtain the so-called Timoshenko equation with respect to the total deflection$w$ given by (Endo [16])

$$\rho A\frac{\partial^2 w}{\partial t^2} + EI\frac{\partial^4 w}{\partial x^4} - \rho I\left(1+\frac{E}{\kappa' G}\right)\frac{\partial^4 w}{\partial x^2\partial t^2} + \frac{\rho^2 I}{\kappa' G}\frac{\partial^4 w}{\partial t^4} = 0 \tag{1}$$

where t is the time, ρ is the density, E and G are respectively Young's and shear moduli, A and I are respectively the cross-sectional area and moment of inertia of area for the beam and k' is the shear correction factor. The same relation as Eq. (1) can be also derived by eliminating the bending deflection w_b from the governing equations for the present beam theory (Endo [16,22]). Similarly, with regard to both of the Mindlin and present plate theories, we can derive the following same equation for w (Mindlin [12]; Endo [16]):

$$\rho h\frac{\partial^2 w}{\partial t^2} + D\nabla^4 w - \rho I_u\left(1+\frac{E}{\kappa' G(1-\nu^2)}\right)\nabla^2\left(\frac{\partial^2 w}{\partial t^2}\right) + \frac{\rho^2 I_u}{\kappa' G}\frac{\partial^4 w}{\partial t^4} = 0, \tag{2}$$

where h is the thickness, ∇^2 is the Laplacian $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$, $\nabla^4 = \nabla^2\nabla^2$is the bi-harmonic operator, ν is the Poisson's ratio, $D = Eh^3/12(1-\nu^2)$ is the plate flexural rigidity and $I_u = h^3/12$. The above mentioned results imply that if we deal with the structural behaviors related to an infinite beam or a plate of infinite extent without boundary constraints the solutions of the present theory would become coincident with those of FSDT, and this may be owing to the fact that "the cross sections of a structural element remain vertical after deformation" if borrowing Donnell's words [17], and therefore the in-plane rotational shear deformations completely vanish.

In the abovementioned context, we may now say that the conditions of a plate or a beam with all boundary edges/ends being simply supported would resemble to those of an infinite plate or beam since restrictiveness on the boundaries are considered to be very loose in the sense that, possibly, no in-plane rotational shear deformations occur.

Considering the above, the main aim of the present paper is to verify that under the boundary conditions with all edges/ends simply supported the theoretical frameworks of the present theory become equivalent to those of the traditional FSDT. In Section 2.1, equivalent property between the Mindlin and present plate theories will be verified, and in Section 2.2 equivalence between the Timoshenko and present beam theories will be shown. Additionally, in Section 3.1, from the practical viewpoint exact natural frequency relationship between the classical Kirchhoff and present/Mindlin plate theories will be obtained for general polygonal plates with all edges simply supported by referring to the works of Wang et al. [27]. Finally, in Section 3.2, simple frequency relationship will be presented based on the series-type synthetic-frequency method (Endo and Taniguchi [28,29]).

2. Equivalent property between the two shear deformable theories

In this section, it will be verified that the present plate and beam theories are equivalent to the Mindlin plate and Timoshenko beam theories, respectively, under the boundary conditions with all edges/ends being simply supported.

2.1. Plate theory

For the Mindlin plate theory [12], three dimensional displacement components u_x, u_y, u_z in a Cartesian co-ordinate system (x, y, z) are assumed as

$$u_x = z\theta_x(x,y,t),\quad u_y = z\theta_y(x,y,t),\quad u_z = w(x,y,t), \tag{3}$$

where θ_x,θ_y are the rotations of the plate element in the $x-z$ and $y-z$ planes. Introducing the above assumption Eq. (3) into three dimensional elasticity relations and based on Hamilton's principle, governing equations in addition to variational consistent boundary conditions are obtained as follows (Endo [22]):

$$\rho h\frac{\partial^2 w}{\partial t^2} - k'Gh\left\{\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}+\theta_x\right) + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}+\theta_y\right)\right\} = q(x,y,t) \quad : \delta w, \tag{4}$$

$$\frac{\rho h^3}{12}\frac{\partial^2\theta_x}{\partial t^2} - D\left(\frac{\partial^2\theta_x}{\partial x^2} + \frac{1+\nu}{2}\frac{\partial^2\theta_y}{\partial x\partial y} + \frac{1-\nu}{2}\frac{\partial^2\theta_x}{\partial y^2}\right) + k'Gh\left(\frac{\partial w}{\partial x}+\theta_x\right) = 0 \quad : \delta\theta_x, \tag{5}$$

$$\frac{\rho h^3}{12}\frac{\partial^2\theta_y}{\partial t^2} - D\left(\frac{\partial^2\theta_y}{\partial y^2} + \frac{1+\nu}{2}\frac{\partial^2\theta_x}{\partial x\partial y} + \frac{1-\nu}{2}\frac{\partial^2\theta_y}{\partial x^2}\right) + k'Gh\left(\frac{\partial w}{\partial y}+\theta_y\right) = 0 \quad : \delta\theta_y, \tag{6}$$