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Consistent limited kinematic hardening plasticity theory and path-independent shakedown theorems

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ABSTRACT

The limited kinematic hardening plasticity theory is re-examined and presented with additional clarifying hypotheses being proposed. The general theory forms the base for construction of the path-independent shakedown theorems intended for possible broad engineering applications, and possible further specifications of particular hardening laws. Some specifications in the literature on limited kinematic hardening are critically analysed. Recent applications of the shakedown theorems in path-independent forms with two separated modes, which are only fully founded in this study, and further problems are briefly summarized with references to the respective literature.

1. Introduction

The classical phenomenological elastic-perfectly plastic theory has been well developed and laid the solid foundation for path-independent (load-history-independent) plastic limit and shakedown theorems [1–10]. While the elastic-perfectly plastic theory and even its extreme case - the rigid-plastic one are sufficiently accurate for plastic limit analysis of engineering structures in applications, the realistic shakedown analysis would require more-refined hardening plasticity modelling. Kinematic hardening plasticity models, starting from Bauschinger effect observation, have been developed to approximate complicated plastic hardening behaviour of materials, which generally is plastic-deformation-path-dependent and can not be described generally and accurately by a single hardening model [1,11–15]. Applications to shakedown analysis have been developed in [1,4,5,16–24]. To reflect the realistic behaviour of materials and to find shakedown limits, the hardening should be limited and some two-yield-surface hardening model has been needed [18,21,23,25–31]. Though the plastic hardening of materials is generally very complex plastic-deformation-path-dependent process, the plastic limit and shakedown analysis in classical spirit should not require all information about the loading and deformation histories, but just the most important core information to predict the path-independent collapse limits in the loading space through the shakedown theorems, as has been pursued in the line of Pham [32]. However the incompleteness of the two-yield-surface hardening modelling raises controversies about the scope of applications and generality of the shakedown formulations and theorems [30,33–35]. The natural question arisen is in which form and under which conditions the shakedown theorems for elastic plastic limited kinematic hardening materials should be path-independent, in compliance with the spirit of classical Melan–Koiter shakedown theorems for elastic perfectly-plastic materials.

In this paper, the consistent limited kinematic hardening plasticity theory with clarifying hypotheses is presented, which sets the framework for the path-independent shakedown theorems and possible further specifications of hardening laws. In Section 2, the basic assumptions and hypotheses of the theory are given. Possible additional specifications are critically analysed in the following section. The Section 4 presents the path-independent shakedown theorems and a brief review of their recent applications. The paper is finished with the discussion and conclusion sections.

2. Basic assumptions and hypotheses

Basic assumptions of classical phenomenological plasticity theory and plastic limit-shakedown theorems include small deformations, plastic incompressibility, the yield stresses in tension and compression being identical, the plastic stress-strain response being rate-independent.

Let $\boldsymbol{\sigma}, \varepsilon^p, \dot{\mathbf{e}}^p$ be the real stress, plastic strain, and plastic strain rate tensors, and $\boldsymbol{\sigma}^*$ be any allowable stress state (i.e. that within the elastic domain inside the yield surface). The plastic deformation is supposed to follow Hill's principle of maximal dissipation:

Maximal dissipation hypothesis

$$(\boldsymbol{\sigma}-\boldsymbol{\sigma}^*) : \dot{\mathbf{e}}^p \geq 0 \quad \text{or} \quad (\boldsymbol{\sigma}-\boldsymbol{\sigma}^*) : d\varepsilon^p \geq 0\,, \tag{1}$$

which implies normality rule (or associated flow law) for the plastic strain rate, and convexity of the yield surface. Stronger Drucker postu-

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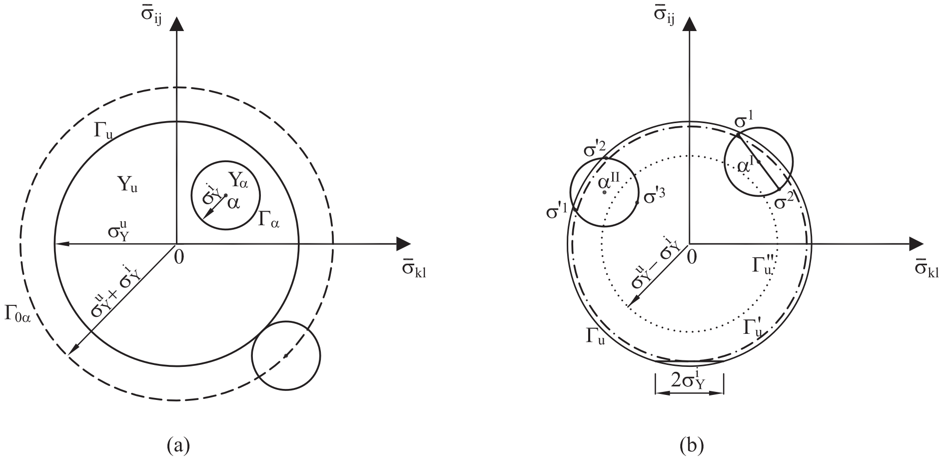

Fig. 1. Yield surfaces in the deviatoric stress coordinates: Γ_u - the ultimate yield surface; Γ_α - the moving inner yield surface centered at $\boldsymbol{\alpha}$ (or $\boldsymbol{\alpha}^I$, $\boldsymbol{\alpha}^{II}$); $\Gamma_{0\alpha}$, Γ'_u, Γ''_u - the possible limiting surfaces for the back stress $\boldsymbol{\alpha}$; $\boldsymbol{\sigma}^1$, $\boldsymbol{\sigma}^2$, or $\boldsymbol{\sigma}'^1$, $\boldsymbol{\sigma}'^2$, $\boldsymbol{\sigma}'^3$ - some stress picks on the inner yield surfaces.

late, which implies Hill principle and requires additionally the material to be stable (softening is not allowed, in particular $d\boldsymbol{\sigma}$: $d\boldsymbol{\varepsilon}^p \geq 0$), can also be assumed. A hypothesis closed to the last stability condition shall be assumed below in inequality (7) [or (13)].

The general limited kinematic hardening is considered. The particular hardening law relating the back stress $\boldsymbol{\alpha}$ to the corresponding plastic deformation $\boldsymbol{\varepsilon}^p_\alpha$ is generally nonlinear, plastic deformation path dependent and need not to be specified, but the imposing hypotheses on the two-surface plastic hardening are stated in this section. Let us consider a representative material element in homogeneous stress-strain state. The size of the yield surface Γ_α, which envelopes the elastic domain $\mathcal{Y}_\alpha$ centred at the (deviatoric) back stress $\boldsymbol{\alpha}$ in the stress space is determined by the initial yield stress σ^i_Y, in particular for Mises material

$$\|\bar{\boldsymbol{\sigma}} - \boldsymbol{\alpha}\|^2_\sigma = \frac{3}{2}(\bar{\boldsymbol{\sigma}} - \boldsymbol{\alpha}) : (\bar{\boldsymbol{\sigma}} - \boldsymbol{\alpha}) = (\sigma^i_Y)^2 \,, \tag{2}$$

where $\bar{\boldsymbol{\sigma}}$ denotes the deviatoric part of the stress tensor $\boldsymbol{\sigma}$. The elasticity domain $\mathcal{Y}_\alpha$ bounded by surface Γ_α is translated in the stress space following its center $\boldsymbol{\alpha}$ without changing size and form. However the hardening is supported to be limited and the set of all allowable stresses is restricted by the unmovable ultimate yield surface Γ_u, which encompasses the respective ultimate domain $\mathcal{Y}_u$ and is defined by the ultimate yield stress σ^u_Y, using the Mises criterium

$$\|\bar{\boldsymbol{\sigma}}\|^2_\sigma = \frac{3}{2}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{\sigma}} = (\sigma^u_Y)^2 \,. \tag{3}$$

A picture of the yield surfaces for the material element (under homogeneous stress-strain state) in the deviatoric stress coordinates $\bar{\sigma}_{ij}$ is presented in Fig. 1a, with the origin of the coordinates being the center of the ultimate yield hypersphere $\mathcal{Y}_u$.

According to the two-yield-surface assumption and the presented picture, the back stress is automatically bounded above by

$$\begin{aligned} &\|\boldsymbol{\alpha}\|_\sigma = \sqrt{\frac{3}{2}}(\boldsymbol{\alpha} : \boldsymbol{\alpha})^{1/2} \leq \|\boldsymbol{\sigma}\|_\sigma + \|\boldsymbol{\alpha} - \boldsymbol{\sigma}\|_\sigma \leq \sigma^u_Y + \sigma^i_Y \,, \\ \text{or} \quad &\|\boldsymbol{\alpha}\| = (\boldsymbol{\alpha} : \boldsymbol{\alpha})^{1/2} \leq \sqrt{\frac{2}{3}}(\sigma^u_Y + \sigma^i_Y) = \hat{\alpha}. \end{aligned} \tag{4}$$

In other words, since the stress $\boldsymbol{\sigma}$ is bounded by the ultimate surface Γ_u, the domain $\mathcal{Y}_\alpha$ containing the recent state $\boldsymbol{\sigma}$ with back stress at the center can not lie entirely outside the domain $\mathcal{Y}_u$. In [32] we gave $\hat{\alpha}$ the smaller value given in Eq. (20) below, coming from the assumption that all the admissible yield surfaces Γ_α are enveloped by the ultimate yield surface Γ_u, which shall be criticized in the next section. In fact the proof of the shakedown theorems requires only $\hat{\alpha}$ to be finite ($\boldsymbol{\alpha}$ is bounded above), not to have any particular value. The surface $\Gamma_{0\alpha}$, under which $\boldsymbol{\alpha}$ should be kept according to Eq. (4), is illustrated in Fig. 1a.

The usual normality yield rule is assumed on both yield surfaces Γ_α and Γ_u, but they do not act simultaneously. When the two criteria attained simultaneously (the stress state is on both yield surfaces), the material yields according to that corresponding to the ultimate yield surface. The total plastic strain rate $\mathbf{e}^p$ and plastic strain $\boldsymbol{\varepsilon}^p$ are composed of those two components:

$$\mathbf{e}^p = \mathbf{e}^p_\alpha + \mathbf{e}^p_u \,, \quad \boldsymbol{\varepsilon}^p = \boldsymbol{\varepsilon}^p_\alpha + \boldsymbol{\varepsilon}^p_u \,. \tag{5}$$

The values $d\sigma_e = \|d\boldsymbol{\sigma}\|_\sigma = \sqrt{\frac{3}{2}d\bar{\boldsymbol{\sigma}} : d\bar{\boldsymbol{\sigma}}}$, $d\alpha_e = \|d\boldsymbol{\alpha}\|_\sigma = \sqrt{\frac{3}{2}d\boldsymbol{\alpha} : d\boldsymbol{\alpha}}$, and $d\varepsilon^p_e = \|d\boldsymbol{\varepsilon}^p\|_\varepsilon = \sqrt{\frac{2}{3}d\boldsymbol{\varepsilon}^p : d\boldsymbol{\varepsilon}^p}$ are called effective stress, back stress, and plastic strain increments, respectively. Under purely uniaxial loading (of a bar), the effective stress is identical to the uniaxial stress, and the effective plastic strain is identical to the uniaxial plastic strain. The stress-strain relation $\sigma - \varepsilon$ and respective relations between σ and ε^p, α and ε^p_α provided from standard uniaxial experiment can be presented as in Fig. 2.

In the classical framework of standard plasticity we have the governing equations

$$\begin{aligned} &\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p_\alpha + \boldsymbol{\varepsilon}^p_u \,, \quad \boldsymbol{\varepsilon}^e = \mathbf{C}^{-1} : \boldsymbol{\sigma} \,, \\ &f_\alpha = \frac{3}{2}(\bar{\boldsymbol{\sigma}} - \boldsymbol{\alpha}) : (\bar{\boldsymbol{\sigma}} - \boldsymbol{\alpha}) - (\sigma^i_Y)^2 \leq 0 \,, \quad \mathbf{e}^p_\alpha = \lambda_\alpha \frac{\partial f_\alpha}{\partial \bar{\boldsymbol{\sigma}}} \\ &f_u = \frac{3}{2}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{\sigma}} - (\sigma^u_Y)^2 \leq 0 \,, \quad \mathbf{e}^p_u = \lambda_u \frac{\partial f_u}{\partial \bar{\boldsymbol{\sigma}}} \,, \end{aligned} \tag{6}$$

where $\boldsymbol{\varepsilon}^e$ and $\mathbf{C}$ are the elastic strain and stiffness tensors.

Pham [32] required that the plastic deformation part $\boldsymbol{\varepsilon}^p_\alpha$, corresponding to the back stress $\boldsymbol{\alpha}$ bounded by (4), is also bounded by some finite value $\hat{\varepsilon}^p_\alpha$. This is in agreement with the long-held views in the literature that unlimited kinematic hardening should not allow incremental collapse, but just the alternating plasticity one, hence $\boldsymbol{\varepsilon}^p_\alpha$ can not increase indefinitely (to be bounded), while the incremental mode is determined by Γ_u and $\boldsymbol{\varepsilon}^p_u$. Instead of such an ad hoc assumption, here we propose a reasonable hypothesis, which implies the limitation of $\boldsymbol{\varepsilon}^p_\alpha$:

Strictly-stable hardening hypothesis

$$d\boldsymbol{\alpha} : d\boldsymbol{\varepsilon}^p_\alpha \geq h_0 d\boldsymbol{\varepsilon}^p_\alpha : d\boldsymbol{\varepsilon}^p_\alpha \,, \tag{7}$$

where h_0 is some non-vanishing positive value.

Indeed, given any possible plastic deformation $\boldsymbol{\varepsilon}^p_\alpha = \tilde{\boldsymbol{\varepsilon}}^p_\alpha$ of a material element, let it deform plastically following the proportional path from $\boldsymbol{\varepsilon}^p_\alpha(0) = \tilde{\boldsymbol{\varepsilon}}^p_\alpha$ to $\boldsymbol{\varepsilon}^p_\alpha(\theta) = \mathbf{0}$

$$\boldsymbol{\varepsilon}^p_\alpha(t) = \left(1 - \frac{t}{\theta}\right)\tilde{\boldsymbol{\varepsilon}}^p_\alpha \,, \quad 0 \leq t \leq \theta. \tag{8}$$